Stability of anyonic superselection sectors

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Topological phases

Quantum phase outside of Landau theory

- ground space degeneracy
- long range entanglement
- anyonic excitations
- modular tensor category / TQFT

Modular tensor category

Describes all properties of the anyons, e.g.
fusion, **braiding**, **charge conjugation**, …

Irreducible objects $\rho_i \Leftrightarrow$ anyons

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How to get the m… …sor category?

Is this stable?

Our approach

- take an **operator algebraic approach**
- ...inspired by **algebraic quantum field theory**
- useful to study **structural questions**
- but also **concrete models**
- can make use of **powerful mathematics**

Quantum phases

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution α_t & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.

Quantum phases of ground states

Two ground states ω_0 and ω_1 are said to be *in the same phase* if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that ω_s is a ground state of $H(s)$.

(Chen, Gu, Wen, *Phys. Rev. B* **82**, 2010)

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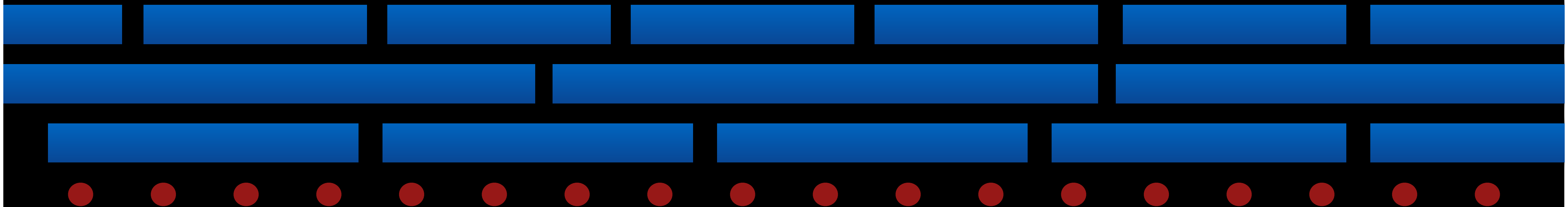
Alternative definition: ω_0 can be transformed into ω_1 with a *finite depth local quantum circuit.*



Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

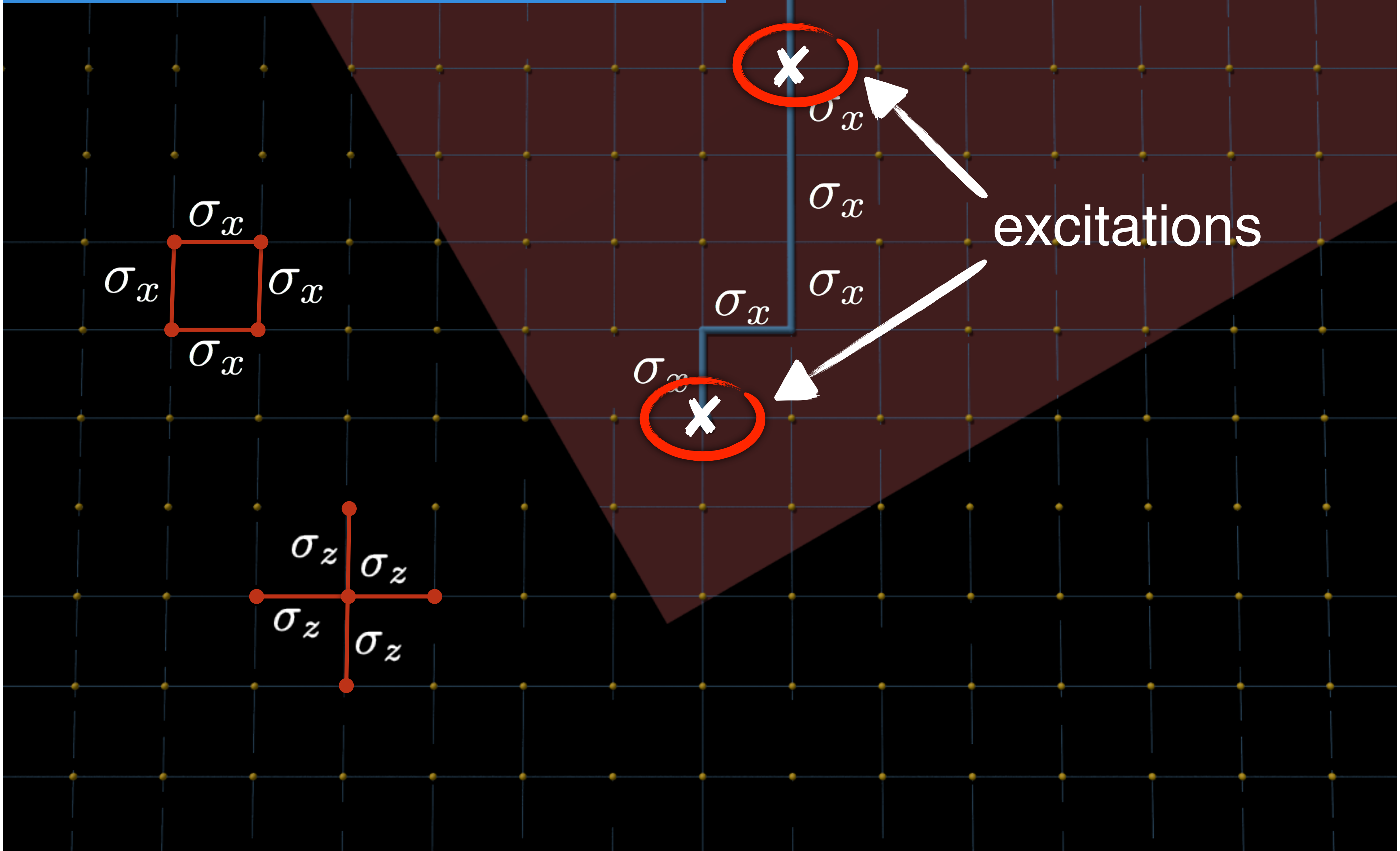
$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

Commun. Math. Phys. **309** (2012)

Moon & Ogata, arXiv:1906:05479 (2019)

Superselection sectors

Example: toric code

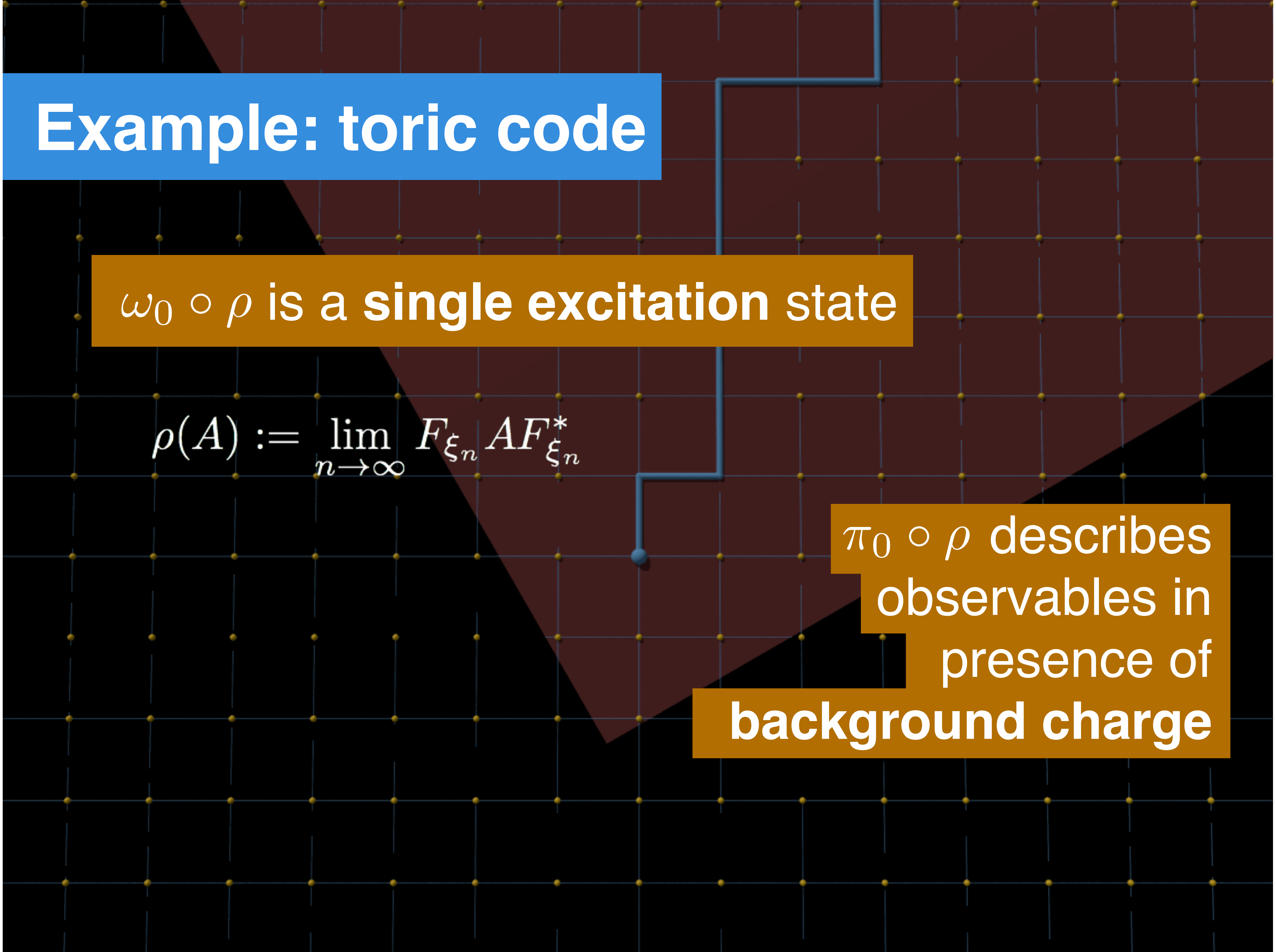


Example: toric code

$\omega_0 \circ \rho$ is a **single excitation** state

$$\rho(A) := \lim_{n \to \infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes observables in presence of **background charge**



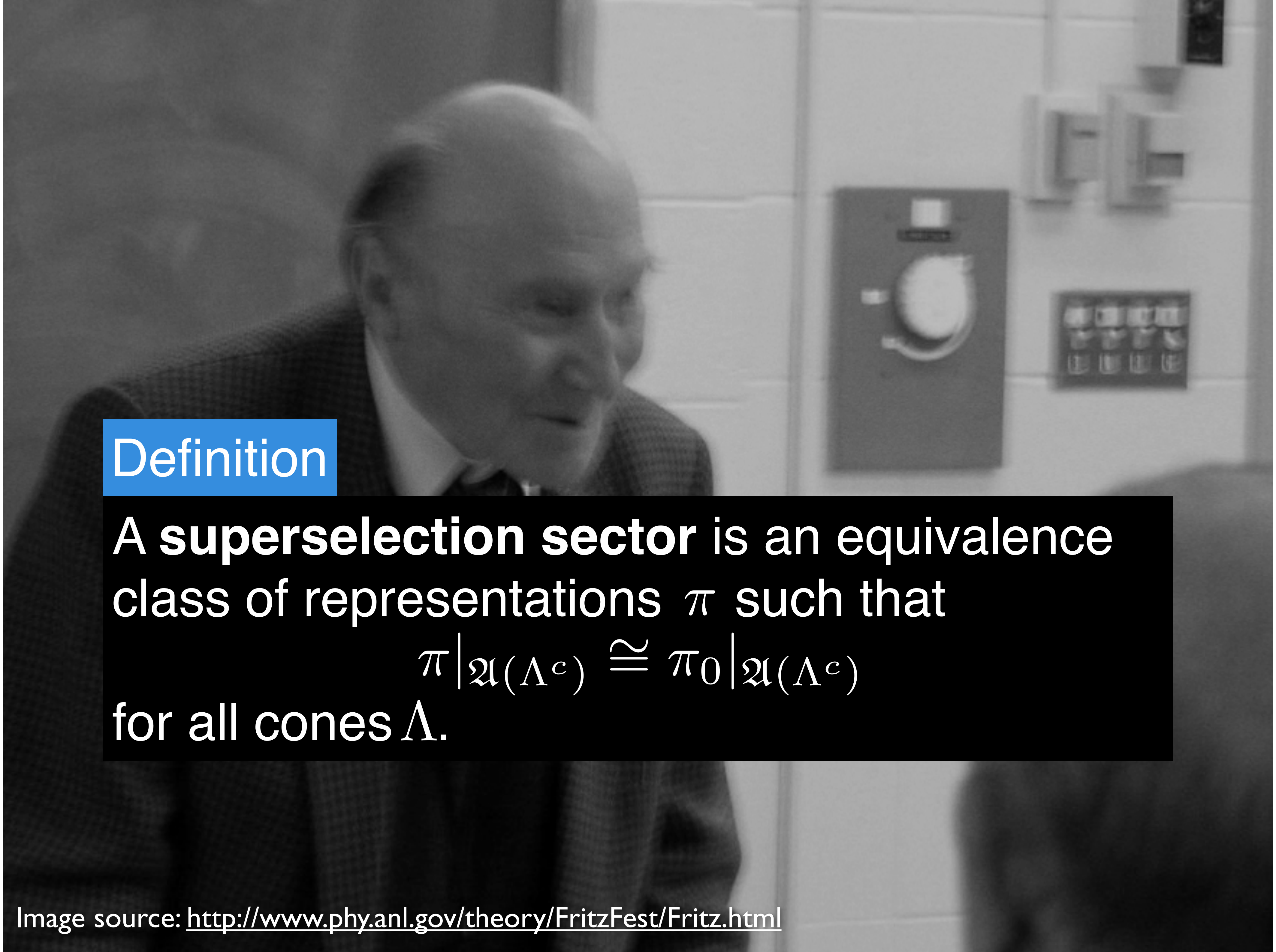
Localised and transportable morphisms

The endomorphism ρ has the following properties:

- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$
- **transportable:** for Λ' there exists σ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these properties (à la Doplicher-Haag-Roberts)

Doplicher, Haag, Roberts, Fredenhagen, Rehren, Schroer, Fröhlich, Gabbiani, …



Definition

A **superselection sector** is an equivalence class of representations π such that

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

for all cones Λ.

Image source: http://www.phy.anl.gov/theory/FritzFest/Fritz.html

Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\mathrm{Rep}\, D(G)$.

Rev. Math. Phys. **23** (2011)
J. Math. Phys. **54** (2013)
Rev. Math. Phys. **27** (2015)

General models

We can obtain a braided tensor category under general conditions:

- **Haag duality:** $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$
- **split property:** $\pi_0(\mathfrak{A}(\Lambda_1))'' \subset \mathfrak{N} \subset \pi_0(\mathfrak{A}(\Lambda_2))''$
- technical property related to direct sums

No reference to Hamiltonian!

Theorem

Let Λ be a cone and suppose that ω_0 is a pure state equivalent to $\omega_\Lambda \otimes \omega_{\Lambda^c}$. Then the corresponding GNS representation π_0 has no non-trivial super selection sectors.

PN, Ogata, *work in progress*

Stability

Stability

How much of the structure is invariant?

- Does the gap stay open under small perturbations?

  Bravyi, Hastings, Michalakis, *J. Math. Phys.* **51** (2010)
  Michalakis, Zwolak, *Commun. Math. Phys.* **322** (2013)

- Is the superselection structure preserved?

  Bravyi, Hastings, Michalakis, *J. Math. Phys.* **51** (2010)
  Haan, *Commun. Math. Phys.* **342** (2016)
  Kato, PN, arXiv:1810.02376

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This is not enough to conclude stability of the superselection structure!

Technical reason

The superselection criterion is defined on the C*-algebraic level…

… but full analysis requires von Neumann algebras (also, split property, Haag duality for π_0)

For example, **intertwiners** $V \in \pi_0(\mathfrak{A}(\Lambda))''$

Not clear if/how α_s extends

Almost localised endomorphisms

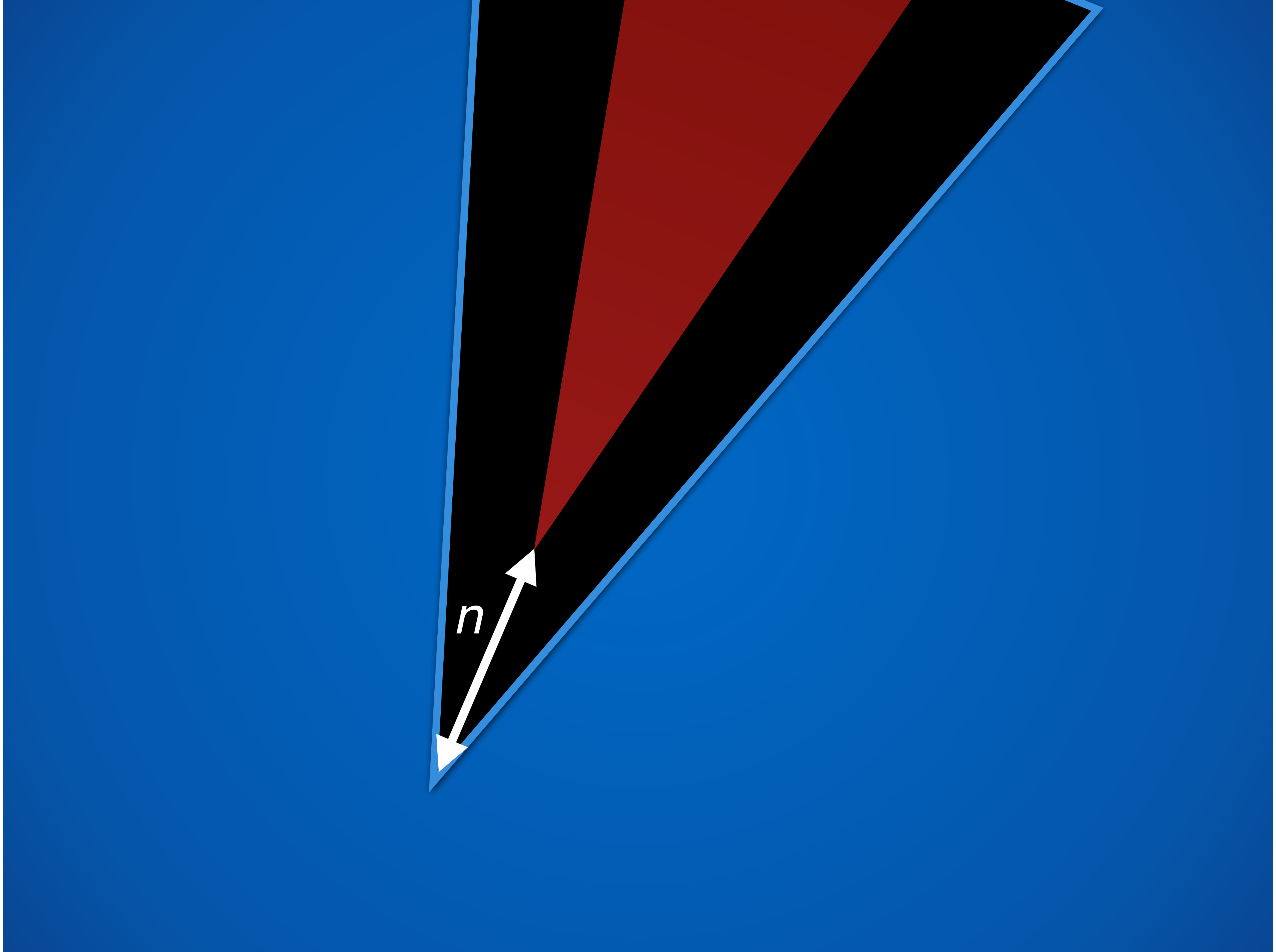
No strict localisation

Almost localised endomorphisms

An endomorphism ρ of $\mathcal{A}$ is called *almost localised* in a cone Λ_α if

$$\sup_{A \in \mathcal{A}(\Lambda^c_{\alpha+\epsilon}+n)} \frac{\|\rho(A) - A\|}{\|A\|} \leq f_\epsilon(n)$$

where $f_\epsilon(n)$ is a non-increasing family of absolutely continuous functions which decay faster than any polynomial in *n*.


n

The semigroup Δ

Define a semigroup Δ of endomorphisms that are

- **almost localised** in cones
- **transportable:** for Λ' there exists σ almost localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$
- intertwiners $(\rho, \sigma)_{\pi_0} := \{T : T\pi_0(\rho(A)) = \pi_0(\sigma(A))T\}$

Can we do sector analysis again?

Stability of Kitaev's quantum double

Almost localised endomorphisms

Follow strategy of Buchholz *et al.*: **asymptopia**

Most tricky part: define tensor structure

$$(\rho \otimes \sigma)(A) := \rho \circ \sigma(A)$$

$$(\rho, \sigma)_\pi := \{T : T\pi(\rho(A)) = \pi(\sigma(A))T\}$$

T in general not in $\mathfrak{A}$! How to define $S \otimes T$?

Intuitively: $S \otimes T = S\rho(T)$

Haag duality is not available!

Buchholz, Doplicher, Morchio, Roberts & Strocchi. In: *Rigorous quantum field theory* (2007)

Asymptotically inner

For general endomorphisms, there are $\{U_n\} \subset \mathfrak{B}(\mathcal{H})$

$$\pi_0(\rho(A)) = \lim_{n\to\infty} U_n \pi_0(A) U_n^*$$

Sequences are not unique, look at such *collections*:

$$\rho(A) = \lim_n U_n A U_n^*, \; \rho'(A) = \lim_n V_n A V_n^*$$

and $R \in (\rho, \rho')_{\pi_0}, \quad R' \in (\sigma, \sigma')_{\pi_0}$ **asymptopia**

$$\lim_{m,n\to\infty} \|[V_n R U_m^*, R']\| = 0$$

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Enough to define fusion

Sec … uch collections:

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Buchholz, Doplicher, Morchio, Roberts & Strocchi. In: *Rigorous quantum field theory* (2007)

Asymptopia

Follow strategy of Buchholz *et al.*: (bi-)**asymptopia**

Using approximate localisation we can get control over the support of $\{U_n\}$

Use this to construct bi-asymptopia and obtain braided tensor category

Buchholz, Doplicher, Morchio, Roberts & Strocchi. In: *Rigorous quantum field theory* (2007)

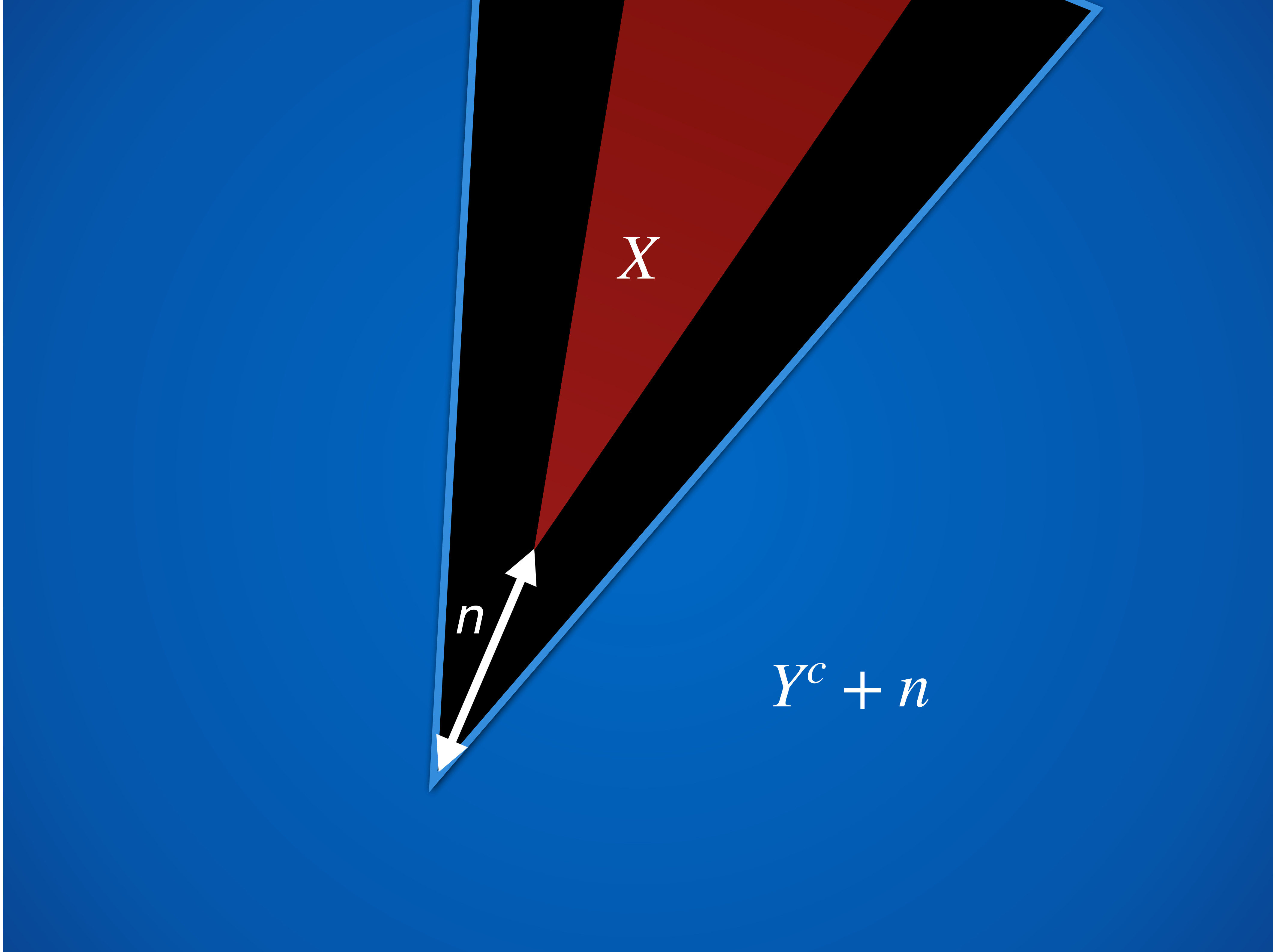
Lieb-Robinson for cones

Quasi-local evolution send observables localised in cones to almost localised observables:

Let X be a cone and Y a cone with a slightly larger opening angle. Then with $A \in \mathfrak{A}(X), B \in \mathfrak{A}(Y^c + n)$

$$\|[\tau_t(A), B]\| \propto \|A\| \|B\| p(d(X, Y + n)) e^{-vt - d(X,Y+n)}$$

Schmitz, Diplomarbeit Albert-Ludwigs-Universität Freiburg (1983)



X
n
Y^c + n

An energy criterion

How are these models related?

Def: write $\mathcal{S}(s)$ for the set of weak-* limits of all states which are mixtures of states with energy < 5. The category $\Delta^{qd}(s)$ consists of all endomorphisms that are:

- almost localised and transportable (wrt. $\omega_0 \circ \alpha_s$)
- $\omega_0 \circ \alpha_s \circ \rho \cong \omega, \quad \omega \in \mathcal{S}(s)$

Putting it all together

- (bi-)asymptopia give braided tensor category $\Delta^{qd}(s)$
- LR bounds give localisation in cones
- can use this to prove $\Delta^{qd}(s) \cong \alpha_s^{-1} \circ \Delta^{qd}(0) \circ \alpha_s$
- unperturbed model is well understood
- need energy criterion

Theorem

Let G be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each s in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\mathrm{Rep}\, D(G)$.

Cha, PN, Nachtergaele, arXiv:1804.03203

Open problems

Non-abelian examples

Energy criterion

When do we get sectors?