Classical capacity of channels between von Neumann algebras

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Infinite quantum systems

Quantum systems with infinitely many d.o.f.:

- Quantum field theory
- Systems in thermodynamic limit…
- e.g. quantum spin systems with topological order

Can we do quantum information?

Infinite quantum systems

$$\mathcal{H} = \mathbb{C}^d \longrightarrow \mathcal{H} = \ell^2(\mathbb{Z}), L^2(\mathbb{R}), \ldots$$

E.g.: infinitely many spins: $\mathcal{H} = \bigotimes_{\mathbb{Z}} \mathbb{C}^2$

~~Stone-von Neumann uniqueness~~

Superselection sectors

Take an operator algebraic approach

Outline

Von Neumann algebras

Quantum information

Example

Von Neumann algebras

Von Neumann algebras

$\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ *-subalgebra and closed in norm

It is a **von Neumann algebra** if closed in w.o.t.:

$$\lim_\lambda \langle \psi, (A - A_\lambda)\psi \rangle = 0 \quad \Rightarrow A \in \mathcal{M}$$

Equivalent definition: $\mathcal{M} = \mathcal{M}''$

A **factor** $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$

$\mathcal{M} \cong \mathfrak{B}(\mathcal{H})$

Can be classified into **Type I**, **Type II**, **Type III**

Normal states

A **state** is a positive linear functional $\omega : \mathcal{M} \to \mathbb{C}$

$$\omega(A^* A) \geq 0, \quad \omega(I) = 1$$

Normal if $\sup_\lambda \omega(X_\lambda) = \omega(\sup_\lambda X_\lambda)$

$$\Leftrightarrow \exists \rho \geq 0 \quad \text{with} \quad \omega(A) = \mathrm{Tr}(\rho A)$$

If a factor $\mathcal{M}$ is not of Type I, there are *no normal pure states*

$$S(\rho) = +\infty$$

Definition of index

For irreducible inclusion $\mathcal{R} \subset \widehat{\mathcal{R}}$

$$\mathcal{E} : \widehat{\mathcal{R}} \to \mathcal{R}, \qquad \mathcal{E}(X^*X) \geq \frac{1}{[\widehat{\mathcal{R}} : \mathcal{R}]} X^*X$$

index is the best constant

Araki relative entropy

Let ω, ϕ be faithful normal states:

Def: $$S_{\varphi,\omega} : x\xi_\varphi \mapsto x^*\xi_\omega$$

$$\Delta(\varphi,\omega) = S_{\varphi,\omega}\overline{S}^*_{\varphi,\omega}$$

Def: $$S(\varphi,\omega) := -\langle \xi_\phi, \log\Delta(\varphi,\omega), \xi_\phi\rangle$$

$$= i \lim_{t\to 0^+} t^{-1}(\varphi([D\omega : D\varphi]_t) - 1)$$

Umegaki: $$S(\rho,\sigma) = \mathrm{Tr}(\rho\log\rho - \rho\log\sigma)$$

Quantum information

- work mainly in the **Heisenberg picture**
- observables modelled by **von Neumann algebra**
- consider **normal states** on $\mathfrak{M}$
- channels are normal unital CP maps $\mathcal{E} : \mathfrak{M} \to \mathfrak{N}$
- Araki **relative entropy** $S(\omega, \phi)$

Quantum information

- use **quantum** systems to communicate
- main question: how much information can I transmit?
- will consider infinite systems here…
- … described by subfactors
- channel capacity is given by Jones index

Quantum information

Sending classical information

$$\mathscr{X} \xrightarrow{\text{encoding}} \bullet \xrightarrow{\text{sending}} \bullet \xrightarrow{\text{decoding}} \mathscr{Y}$$

State preparation

$$x \mapsto \omega_x \mapsto \mathscr{E}^*(\omega_x) := \omega_x \circ \mathscr{E}$$

POVM measurement

$$E_y \geq 0, \quad \sum_{y \in \mathscr{Y}} E_y = I$$

Gives a classical channel $\mathscr{X} \to \mathscr{Y}$!

Shannon information $I(X : Y)$

Distinguishing states

Alice prepares a mixed state ρ:

$$\rho = \sum_{i=1}^{n} p_i \rho_i$$

…and sends it to Bob

Can Bob recover $\{p_i\}$?

Holevo χ quantity

In general not exactly:

$$\chi(\{p_i\},\{\rho_i\}) := S(\rho) - \sum_i p_i S(\rho_i)$$

$$= \sum_i p_i S(\rho_i, \rho)$$

Generalisation of **Shannon information**

Holevo χ quantity

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$$\chi(\{p_i\}, \{\rho_i\}) := S(\rho) - \sum_i p_i S(\rho_i)$$

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Generalisation of **Shannon information**

Infinite systems

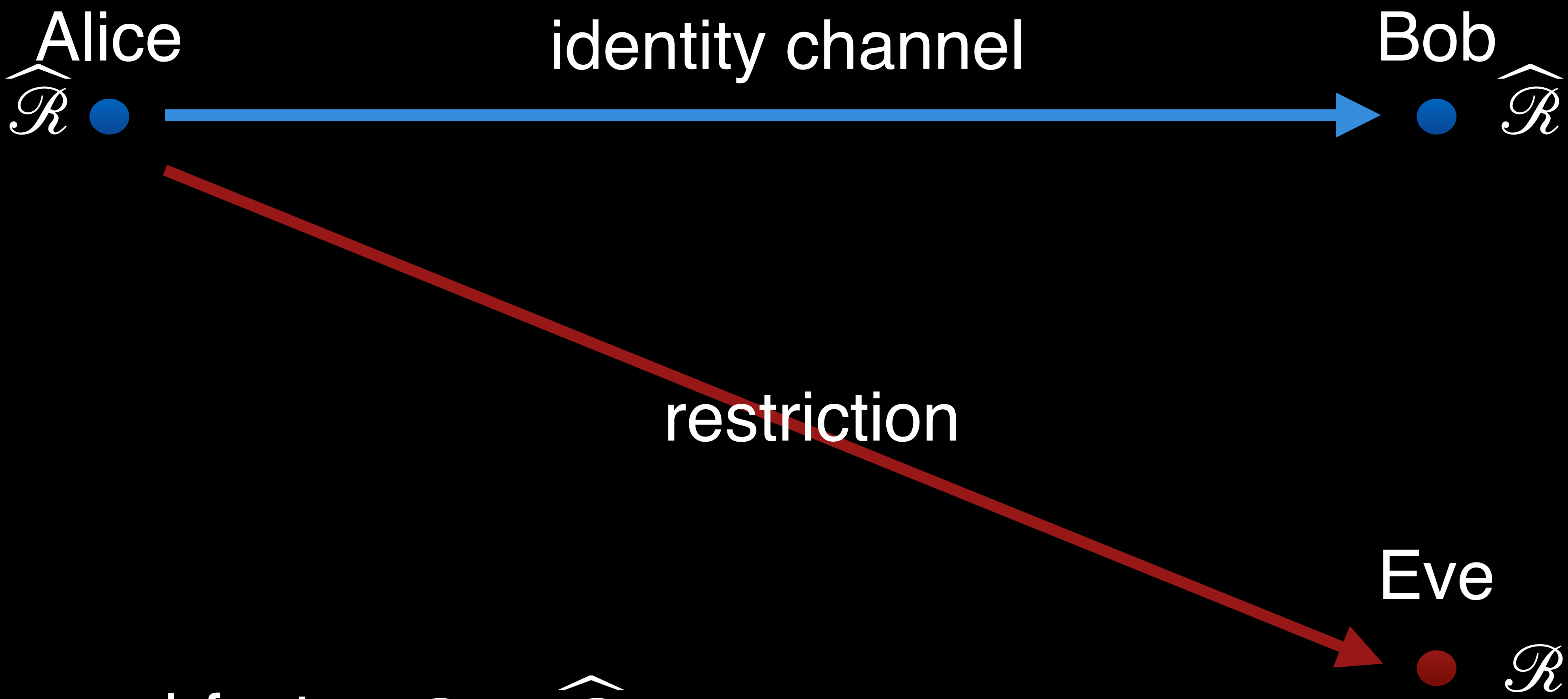
Suppose $\mathfrak{M}$ is an infinite factor, say Type III, and φ a faithful normal state

$$\sup_{(\varphi_i)} \sum p_x S(\varphi_x, \varphi) = \infty$$

where $\varphi = \sum_x p_x \varphi_x$

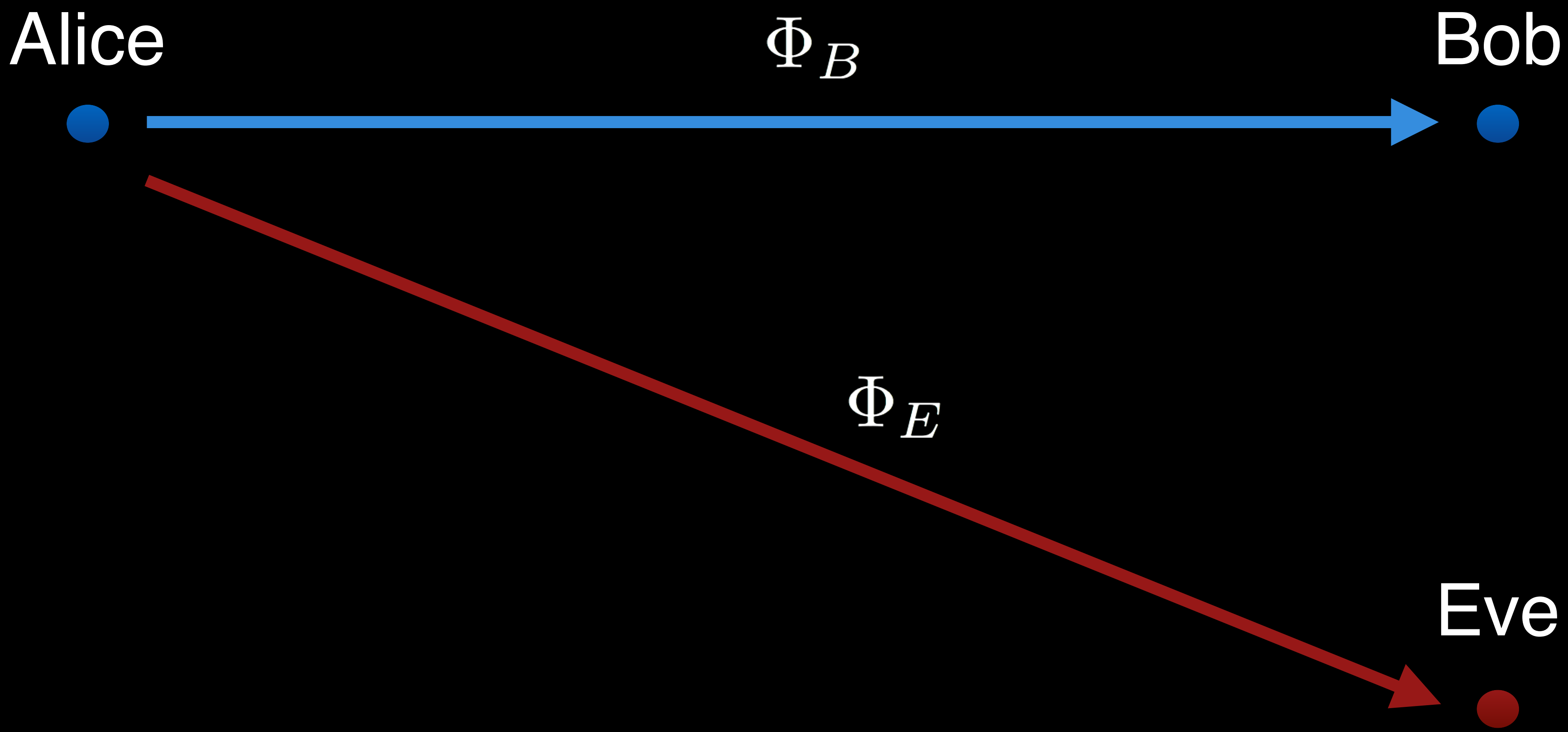
Better to compare algebras!

Limited access



subfactor $\mathscr{R} \subset \widehat{\mathscr{R}}$

Quantum wiretapping



Theorem (Devetak, Cai/Winter/Yeung)

The rate of a wiretapping channel is given by

$$\lim_{n\to\infty} \frac{1}{n} \max_{\{p_x,\rho_x\}} \left(\chi(\{p_x\},\Phi_B^{\otimes n}(\rho_x)\}) - \chi(\{p_x\},\Phi_E^{\otimes n}(\rho_x)\})\right)$$

Comparing algebras

Want to compare $\widehat{\mathcal{R}}$ and $\mathcal{R}$, with $\mathcal{R} \subset \widehat{\mathcal{R}}$ subfactor

$$H_\phi(\widehat{\mathcal{R}}|\mathcal{R}) = \sup_{(\phi_i)} \left(\sum_i [S(p_i\phi_i, \phi) - S(p_i\phi_i \restriction \mathcal{R}, \phi \restriction \mathcal{R})] \right)$$

$$= \sup_{(\phi_i)} (\chi(\{p_i\}, \{\phi_i\}) - \chi(\{p_i\}, \{\phi_i \restriction \mathcal{R}\}))$$

Δ_χ

entropic disturbance

Jones index and entropy

constrained channel

$$\log[\widehat{\mathcal{R}} : \mathcal{R}] = \sup_{\phi:\phi\circ\mathcal{E}=\phi} H_\phi(\widehat{\mathcal{R}}|\mathcal{R})$$

Hiai, J. Operator Theory, ’90; J. Math. Soc. Japan, ‘91

gives an **information-theoretic** interpretation to the Jones index

Single-letter formula

$$\lim_{n\to\infty} \frac{1}{n} \sup_{\{p_x\},\{\omega_x\}} \chi(\{p_x\},\{\omega_x\}) - \chi(\{p_x\},\{\omega_x | \mathscr{R}^{\otimes n}\})$$

states on $\widehat{\mathscr{R}}^{\otimes n}$ with $\omega \circ \mathscr{E}^{\otimes n} = \omega$

$$= \lim_{n\to\infty} \frac{1}{n} \log[\widehat{\mathscr{R}}^{\otimes n} : \mathscr{R}^{\otimes n}]$$

It can be shown that $[\widehat{\mathscr{R}}^{\otimes n} : \mathscr{R}^{\otimes n}] = [\widehat{\mathscr{R}} : \mathscr{R}]^n$

Hence we get a **single letter formula**!

What is missing?

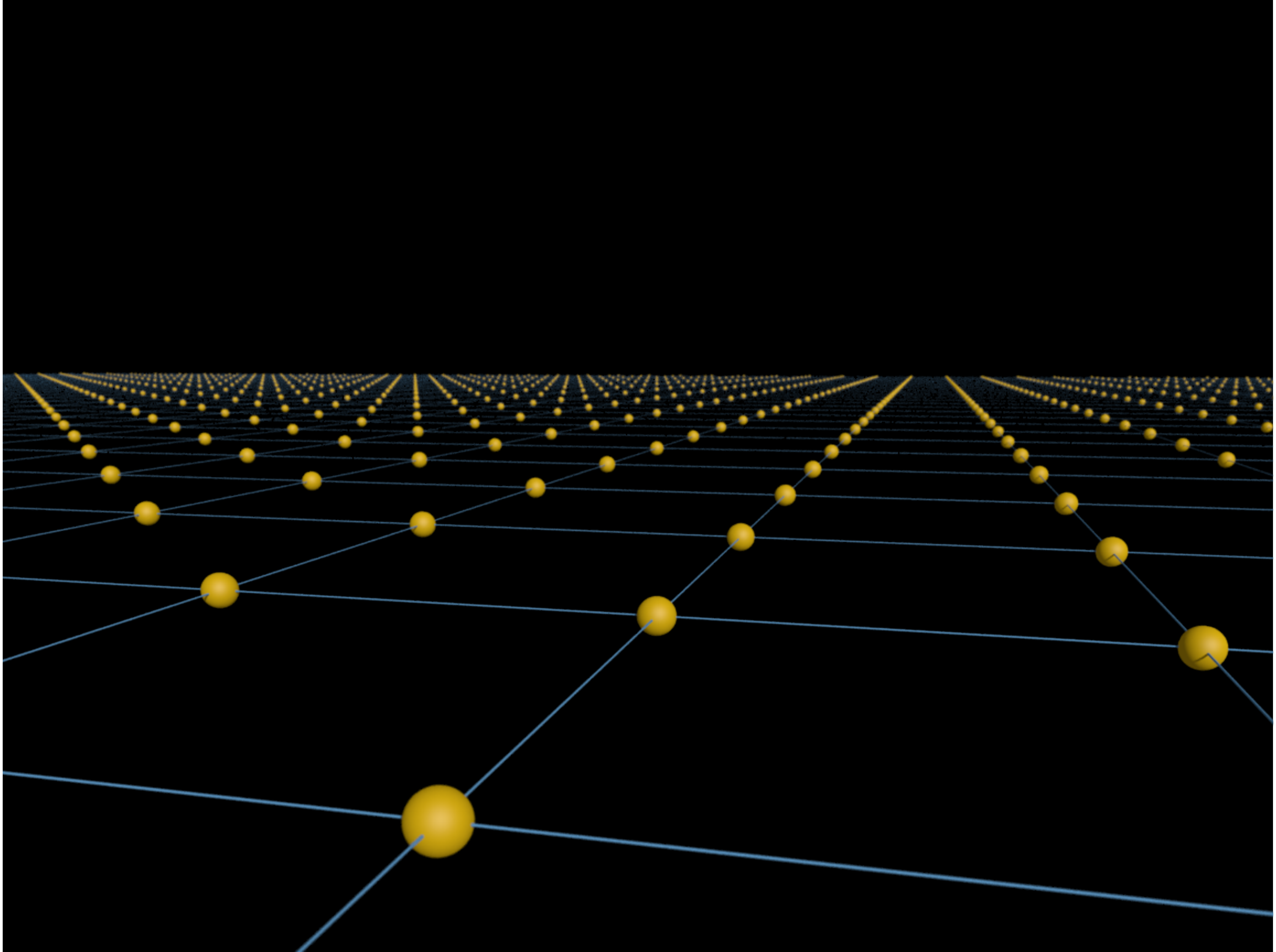
No **coding theorem** yet!

- no pure states
- analogue of typical subspaces?
- look at hyperfinite factors?

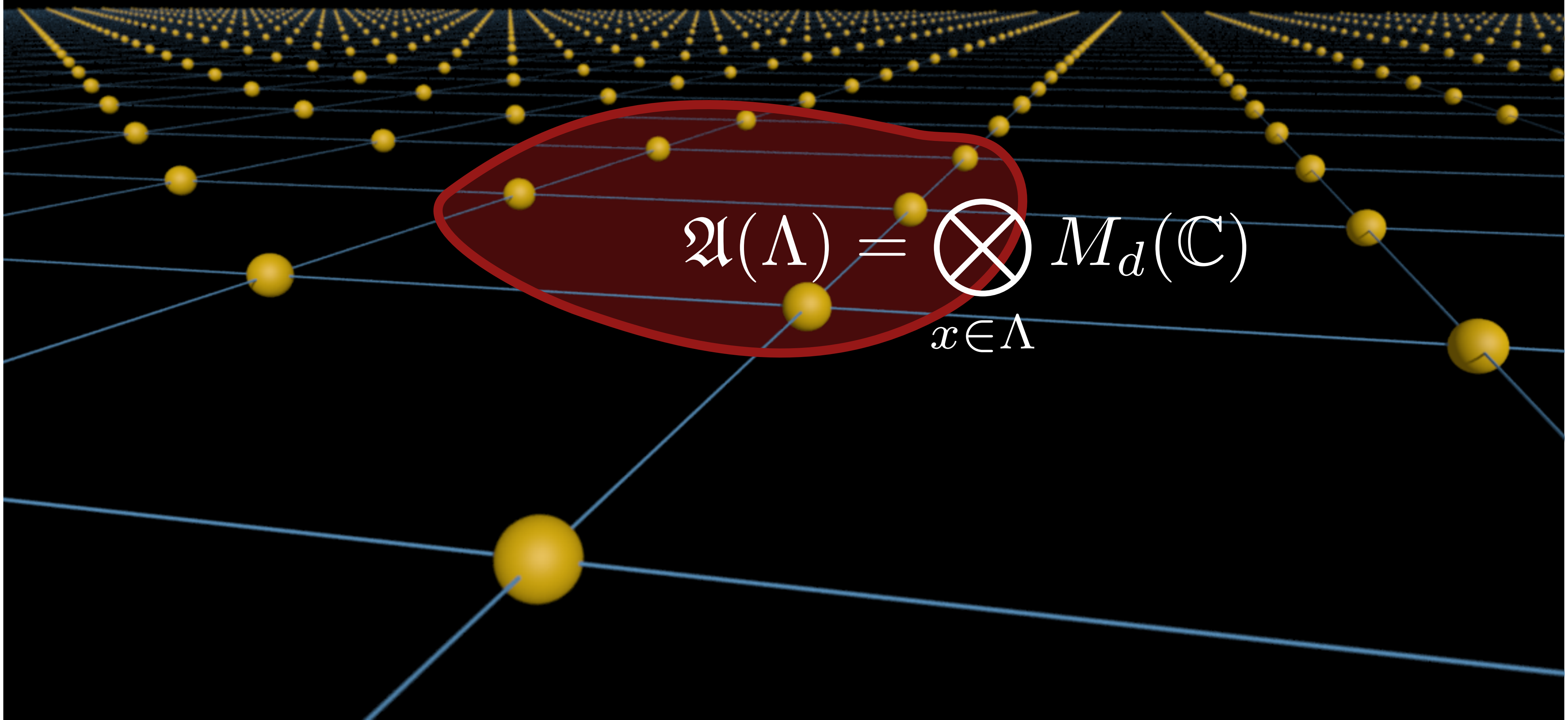
Positive side:

- can find states in concrete examples
- subfactors are well studied

Example

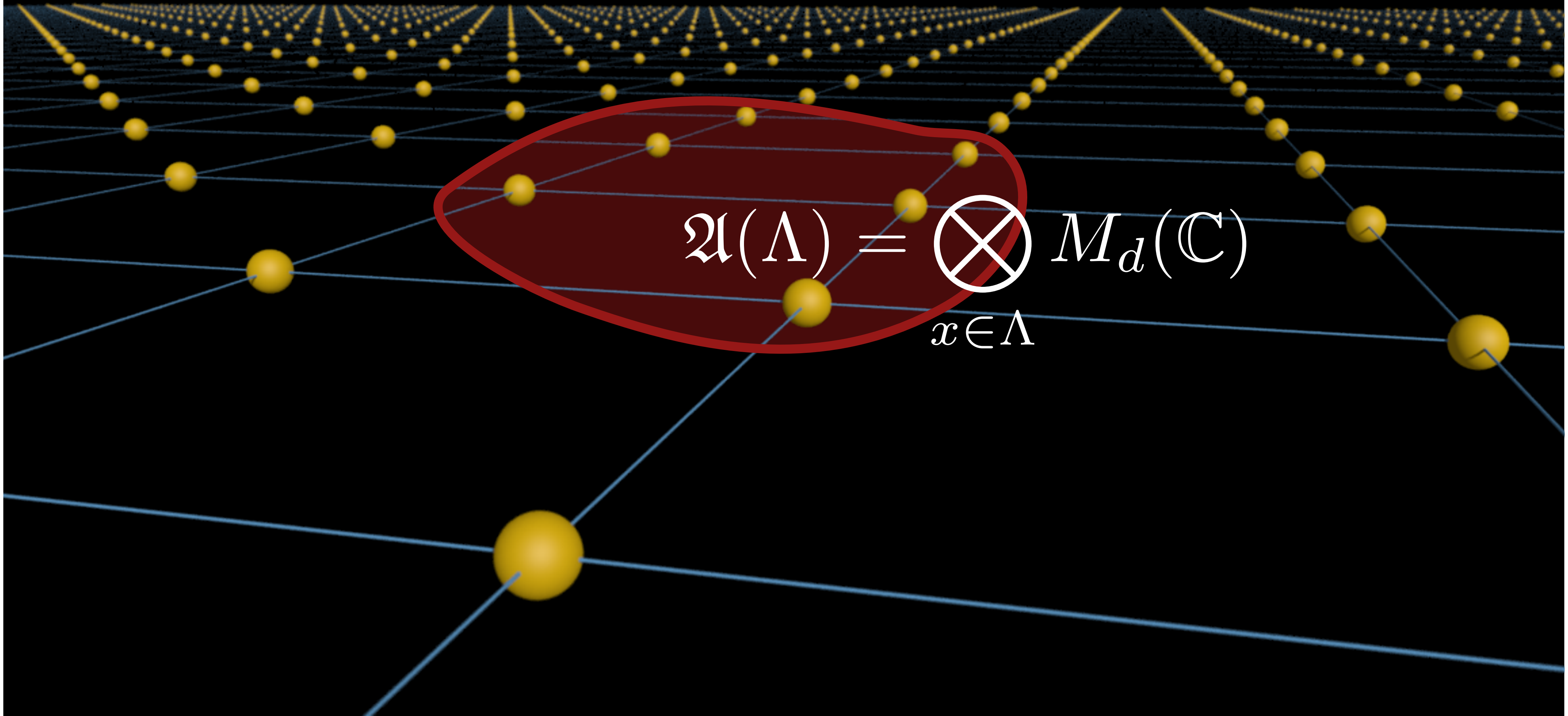


Quasi-local algebra $\mathfrak{A} = \overline{\bigcup_{\Lambda} \mathfrak{A}(\Lambda)}^{\|\cdot\|}$



$$\mathfrak{A}(\Lambda) = \bigotimes_{x \in \Lambda} M_d(\mathbb{C})$$

and **local Hamiltonians** $H_\Lambda \in \mathfrak{A}(\Lambda)$



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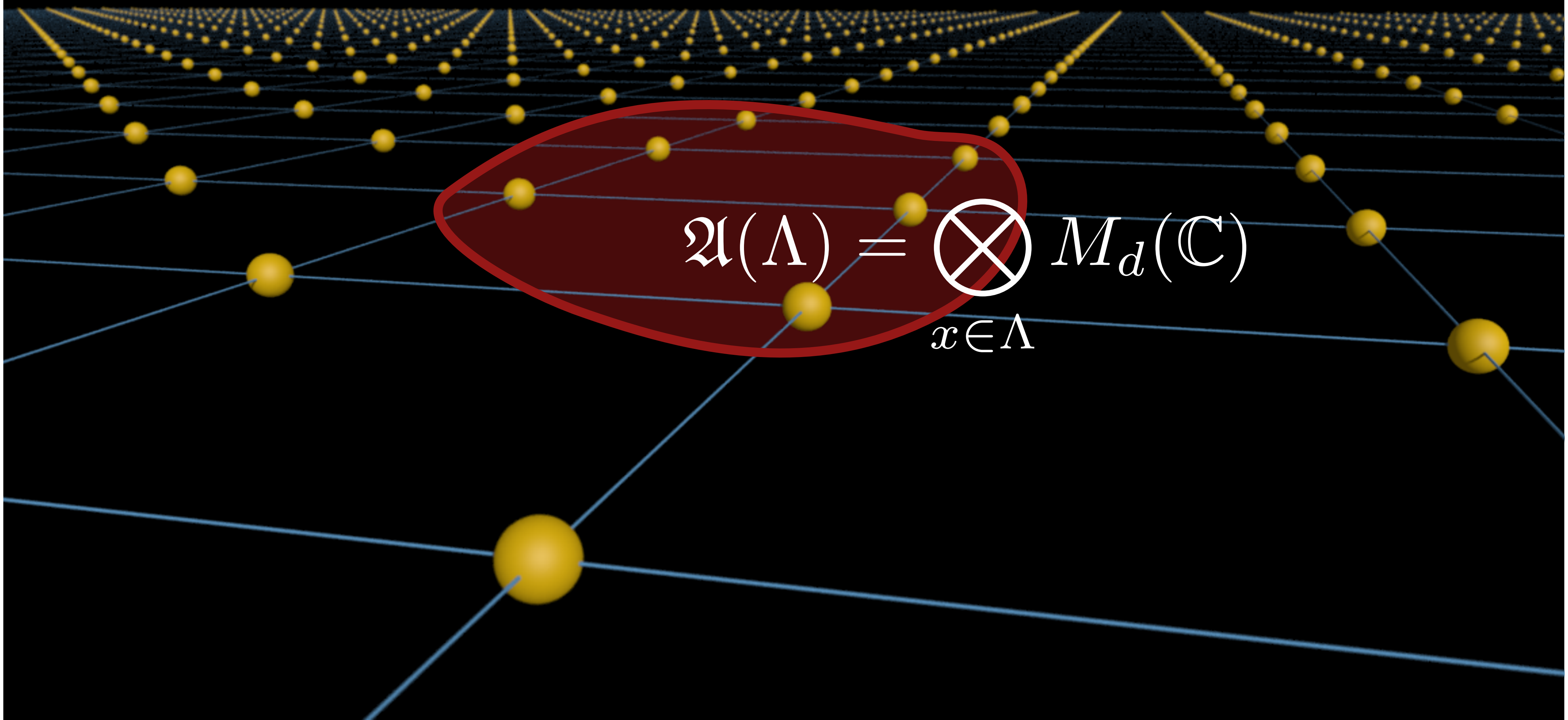
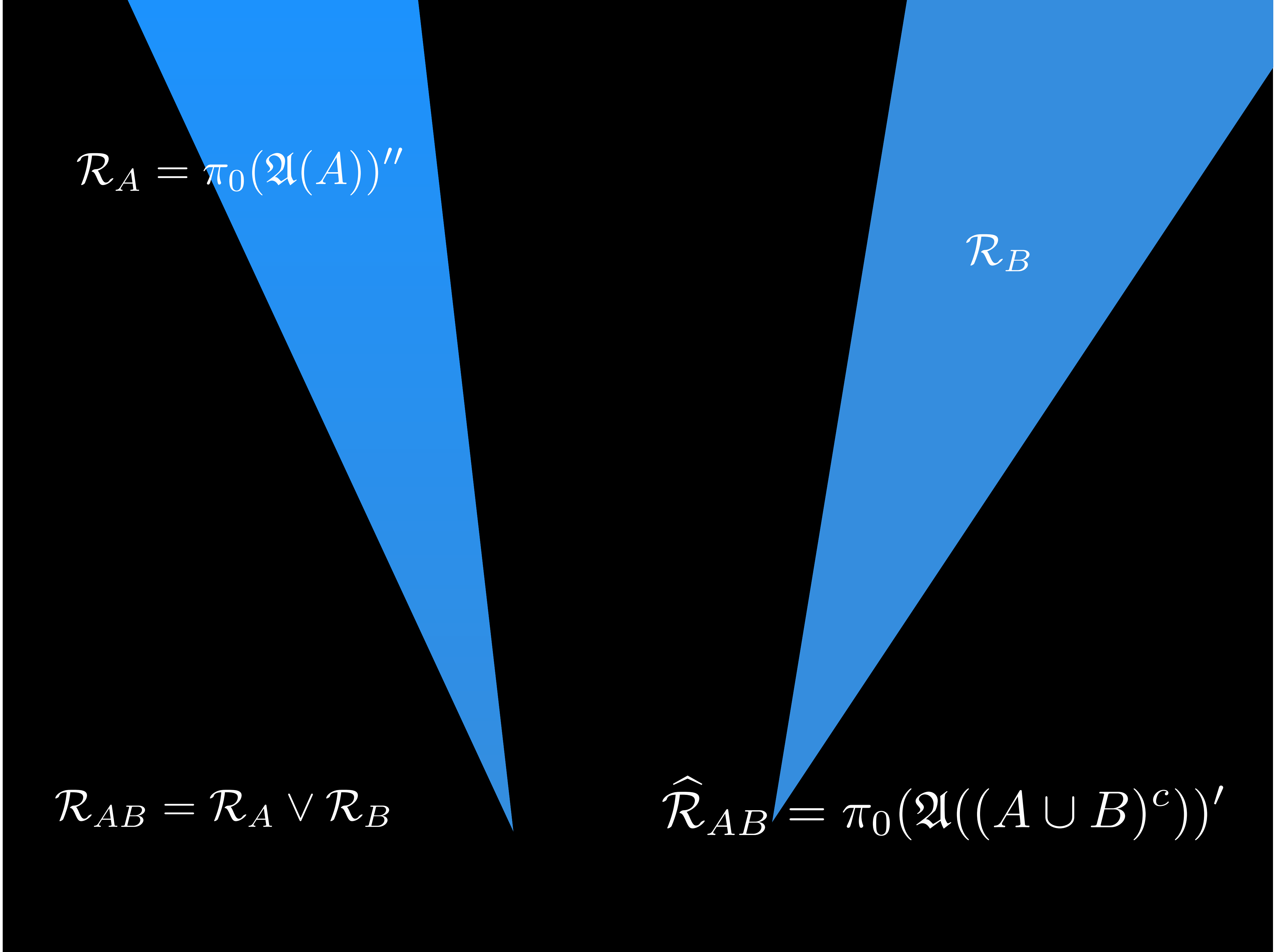
ground state representation π_0



$$\mathfrak{A}(\Lambda) = \bigotimes_{x \in \Lambda} M_d(\mathbb{C})$$

Toric code

- unique **translation invariant** ground state ω_0
- corresponding GNS representation π_0
- can identify anyonic excitations with $\pi_0 \circ \rho$
- where ρ is localised and transportable autom.
- can recover all anyons and their properties

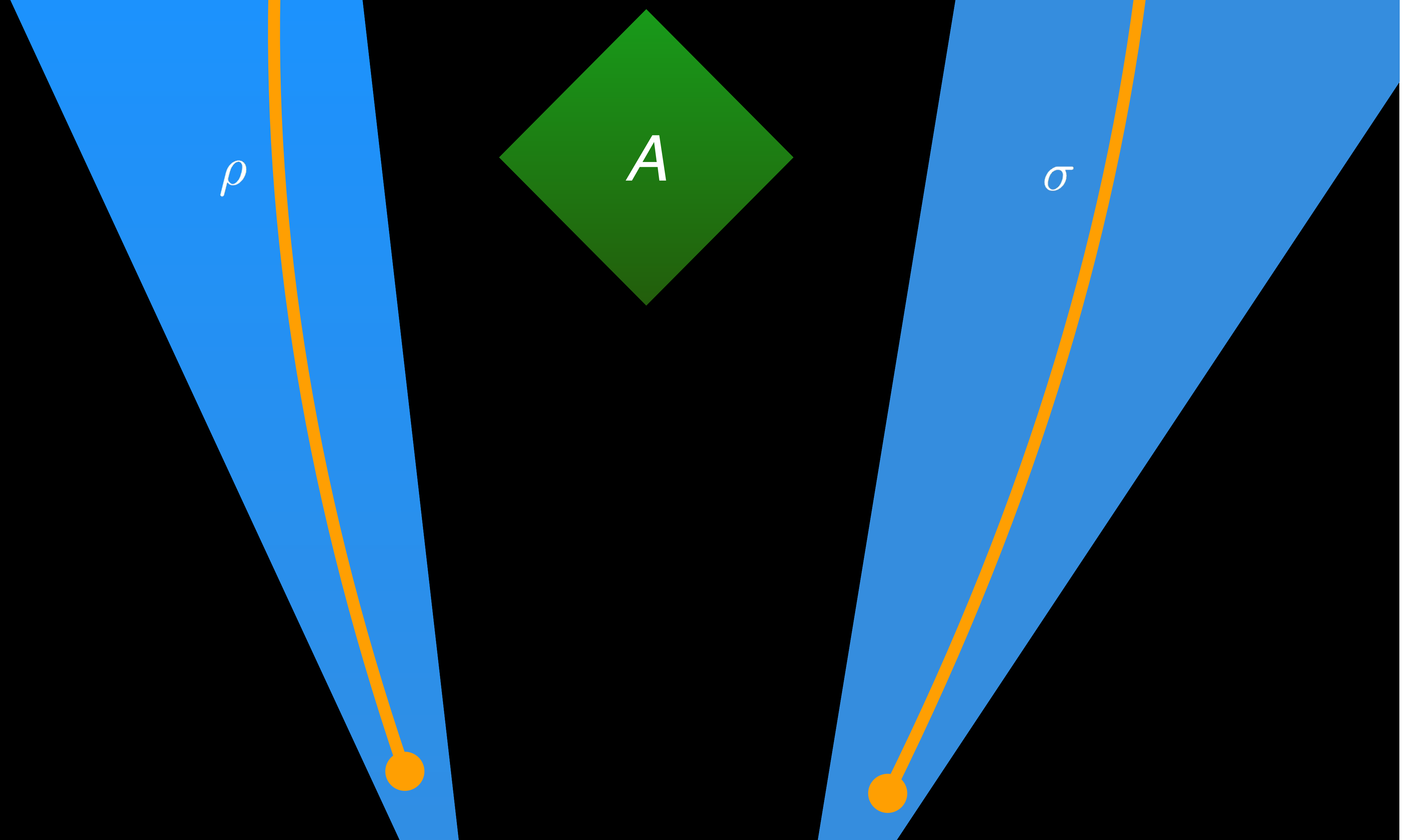

$\mathcal{R}_A = \pi_0(\mathfrak{A}(A))''$
$\mathcal{R}_B$
$\mathcal{R}_{AB} = \mathcal{R}_A \vee \mathcal{R}_B$
$\widehat{\mathcal{R}}_{AB} = \pi_0(\mathfrak{A}((A \cup B)^c))'$

Locality: $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$

but:

$$\mathcal{R}_{AB} \subsetneqq \widehat{\mathcal{R}}_{AB}$$

$$[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}] = \sum_i d_i^2$$



$$V\pi_0(A) = V\pi_0(\rho(A)) = \pi_0(\sigma(A))V = \pi_0(A)V$$

$$\Rightarrow V \in \pi_0(\mathfrak{A}((A \cup B)^c))' = \widehat{\mathcal{R}}_{AB}$$

Some remarks

Four states that can be **distinguished perfectly** on $\widehat{\mathscr{R}}_{AB}$…

… but coincide on $\mathscr{R}_{AB}$

Inclusion of finite dim. algebras

$A \mapsto A \oplus A \oplus A \oplus A \ldots$

with "convergence" to $\mathscr{R}_{AB} \subset \widehat{\mathscr{R}}_{AB}$

Conclusions

Conclusions

QI channels for op. algs.

Coding theorem missing

Stability of capacity?