Capacity of quantum channels from subfactors

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Infinite quantum systems

Quantum systems with infinitely many d.o.f.:

- Quantum field theory
- Systems in thermodynamic limit…
- e.g. quantum spin systems with topological order

Can we do quantum information?

Infinite quantum systems

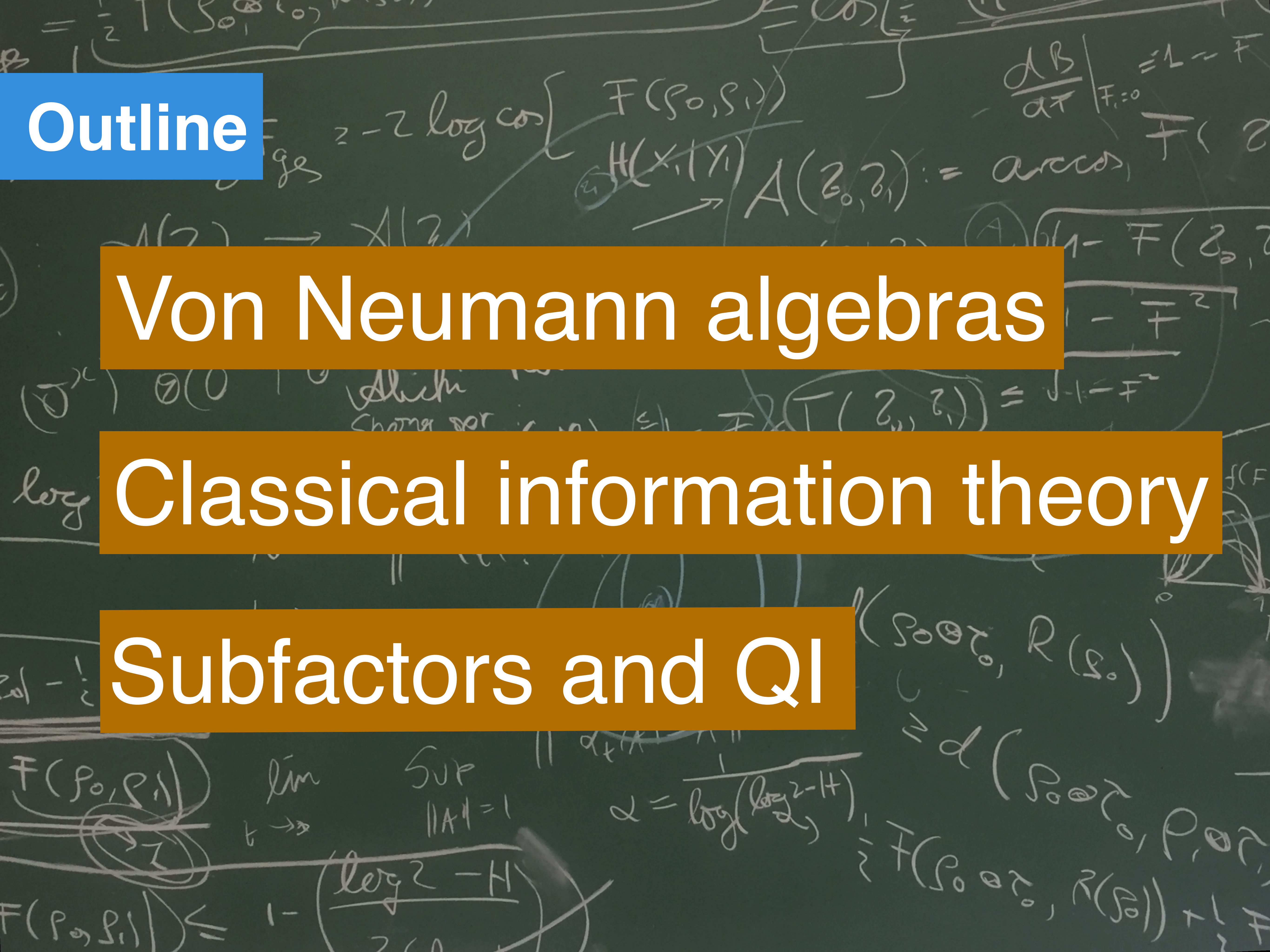
$$\mathcal{H} = \cancel{\mathbb{C}^d} \longrightarrow \mathcal{H} = \ell^2(\mathbb{Z}), L^2(\mathbb{R}), \dots$$

E.g.: infinitely many spins: $\mathcal{H} = \bigotimes_{\cancel{\mathbb{Z}}} \mathbb{C}^2$

~~Stone-von Neumann uniqueness~~

Superselection sectors

Take an operator algebraic approach



Outline

Von Neumann algebras

Classical information theory

Subfactors and QI

Von Neumann algebras

Von Neumann algebras

$\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ *-subalgebra and closed in norm

It is a **von Neumann algebra** if closed in w.o.t.:

$$\lim_\lambda \langle \psi, (A - A_\lambda)\psi \rangle = 0 \quad \Rightarrow A \in \mathcal{M}$$

Equivalent definition: $\mathcal{M} = \mathcal{M}''$

A **factor** $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$

$\mathcal{M} \cong \mathfrak{B}(\mathcal{H})$

Can be classified into **Type I**, **Type II**, **Type III**

Normal states

A **state** is a positive linear functional $\omega : \mathcal{M} \to \mathbb{C}$

$$\omega(A^* A) \geq 0, \quad \omega(I) = 1$$

Normal if $\sup_\lambda \omega(X_\lambda) = \omega(\sup_\lambda X_\lambda)$

$\Leftrightarrow \exists \rho \geq 0$ with $\omega(A) = \mathrm{Tr}(\rho A)$

If a factor $\mathcal{M}$ is not of Type I, there are *no normal pure states*

$$S(\rho) = +\infty$$

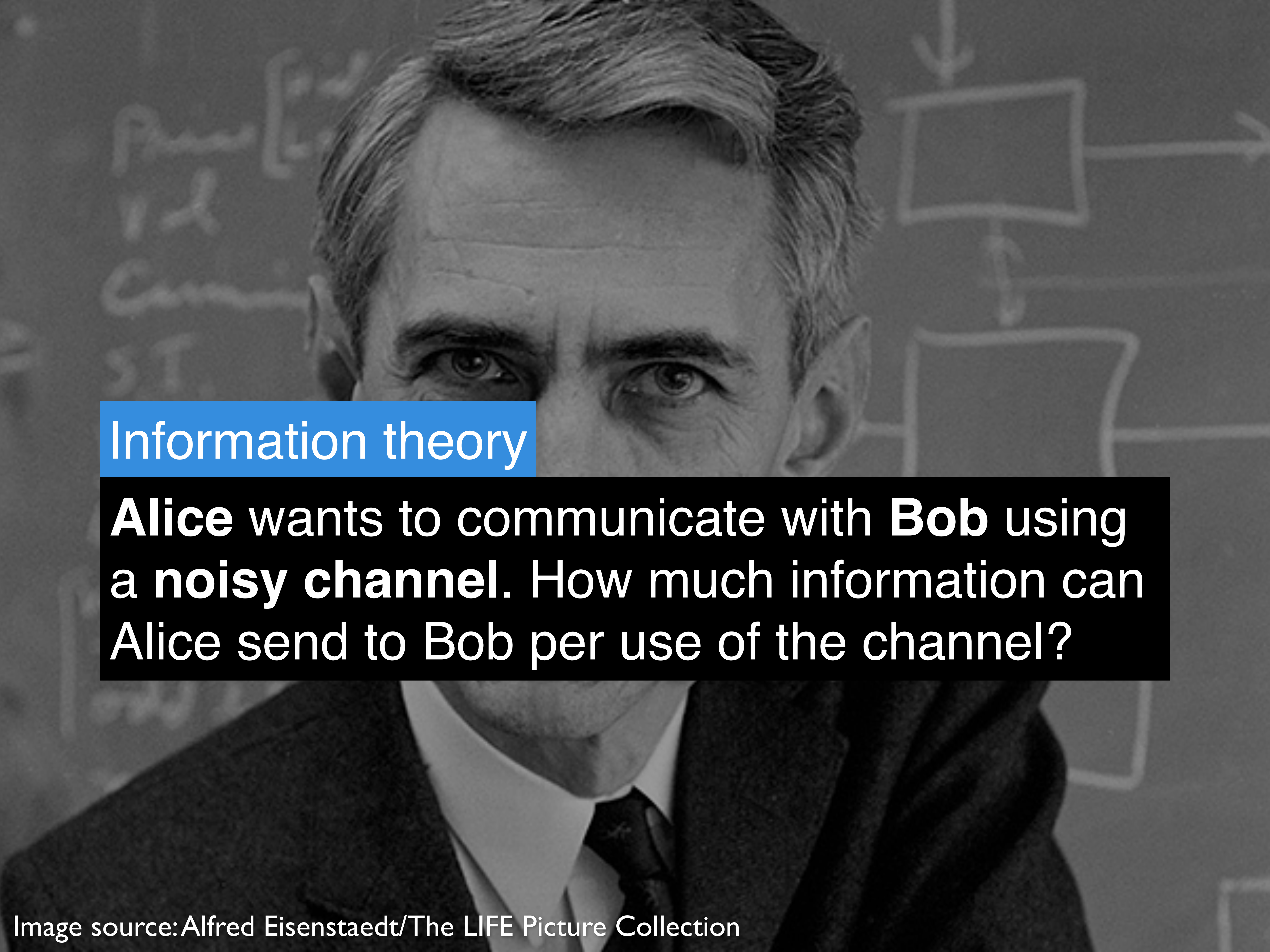
Quantum information

- work mainly in the **Heisenberg picture**
- observables modelled by **von Neumann algebra**
- consider **normal states** on $\mathfrak{M}$
- channels are normal unital CP maps $\mathcal{E} : \mathfrak{M} \to \mathfrak{N}$
- Araki **relative entropy** $S(\omega, \phi)$

Quantum information

- use **quantum** systems to communicate
- main question: how much information can I transmit?
- will consider infinite systems here…
- … described by subfactors
- channel capacity is given by Jones index

Classical wiretapping channels



Information theory

Alice wants to communicate with **Bob** using a **noisy channel**. How much information can Alice send to Bob per use of the channel?

Image source: Alfred Eisenstaedt/The LIFE Picture Collection

Setup

Alice $\xrightarrow{p(y|x)}$ Bob

$\mathcal{X}$ input space

$\{p_x\}_{x\in\mathcal{X}}$

$\mathcal{Y}$ output space

$$p_y = \sum_{x\in\mathcal{X}} p(y|x)$$

How well can Bob recover the messages sent by Alice (small error allowed)?

Relative entropy

Compare two probability distributions *P*, *Q*:

$$H(P:Q) = \begin{cases} \sum p_x \log \frac{p_x}{q_x} & \{x : p_x > 0\} \subset \{x : q_x > 0\} \\ +\infty & \text{else} \end{cases}$$

Vanishes iff *P*=*Q*, otherwise positive

Mutual information

`information' due to noise

$$I(X:Y) = H(Y) - H(X|Y)$$

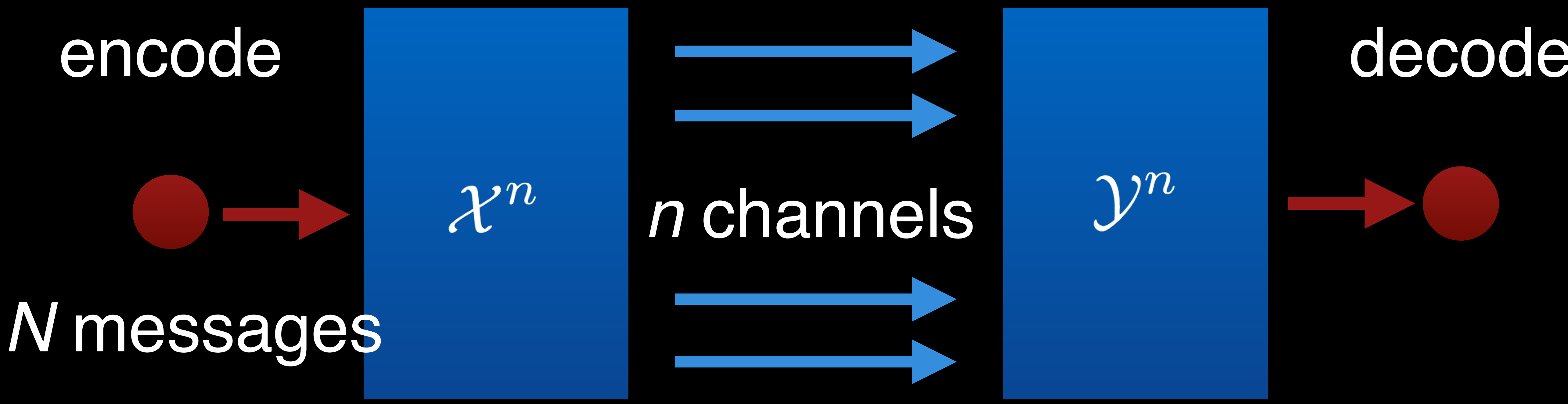
here the conditional entropy is defined:

$$H(Y|X) = \sum_x p_x H(Y|X=x)$$

some algebra gives: $P'_x = \{p(y|x)\}$ $\quad P' = \sum_x p_x P'_x$

$$I(X:Y) = \sum_x p_x H(P'_x : P')$$

Operational approach



Maximum error for *all* possible encodings:

$$p_e(n, N)$$

Coding theorem

Def: *R* is called an *achievable rate* if

$$\lim_{n\to\infty} p_e(n, 2^{nR}) = 0$$

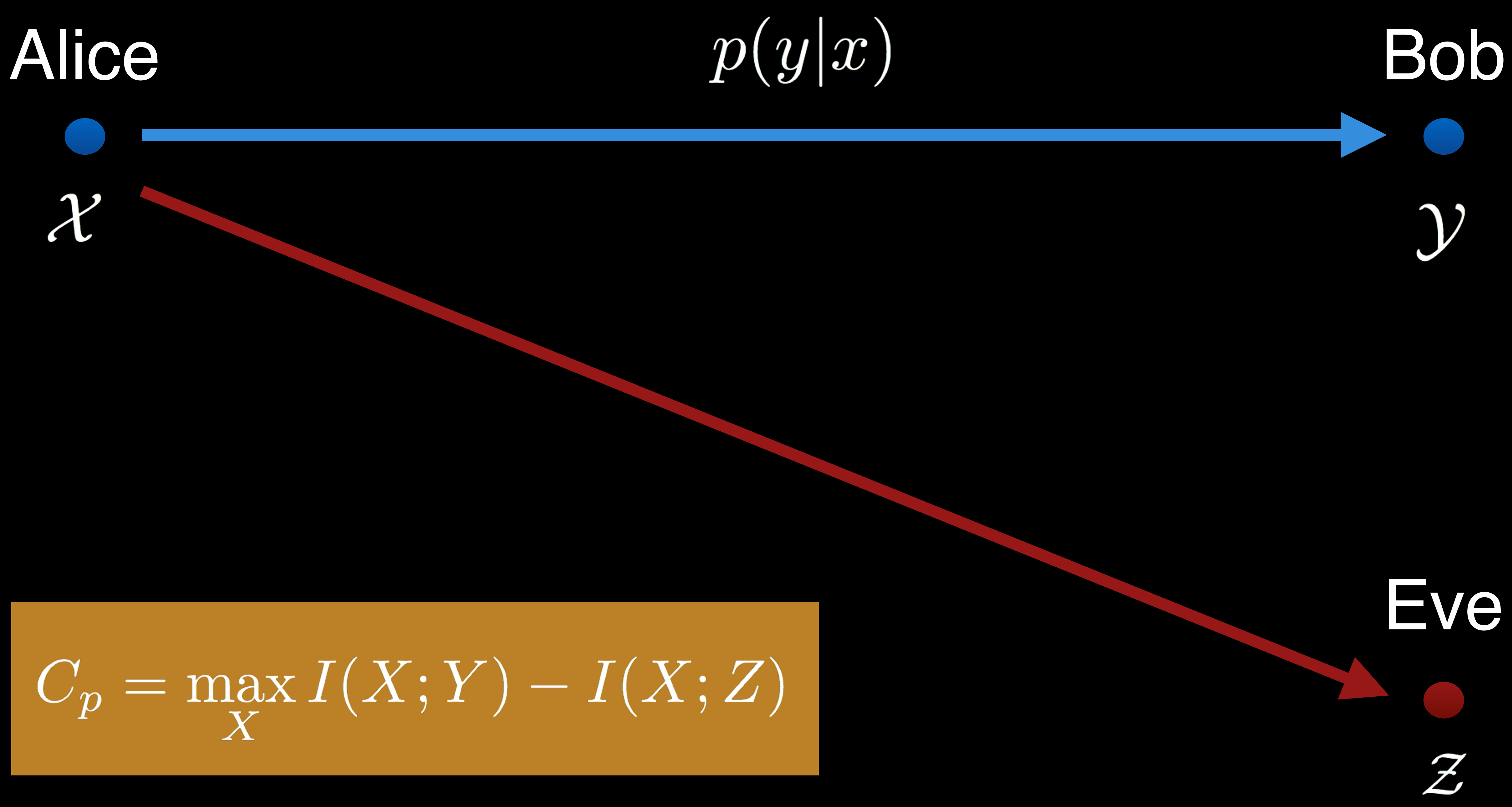
The supremum of all *R* is called the **capacity** *C*.

Theorem: the capacity is the *Shannon capacity* of the channel, defined as:

$$C_{Shan} = \max_X I(X:Y)$$

This is a single-letter formula!

Wiretapping channels



$$C_p = \max_X I(X;Y) - I(X;Z)$$

Quantum information

Distinguishing states

Alice prepares a mixed state ρ:

$$\rho = \sum_{i=1}^{n} p_i \rho_i$$

…and sends it to Bob

Can Bob recover $\{p_i\}$?

Holevo χ quantity

In general not exactly:

$$\chi(\{p_i\},\{\rho_i\}) := S(\rho) - \sum_i p_i S(\rho_i)$$
$$= \sum_i p_i S(\rho_i, \rho)$$

Generalisation of **Shannon information**

Araki relative entropy

Let ω, ϕ be faithful normal states:

Def:

$$S_{\varphi,\omega} : x\xi_\varphi \mapsto x^*\xi_\omega$$

$$\Delta(\varphi,\omega) = S_{\varphi,\omega}\overline{S}^*_{\varphi,\omega}$$

Def:

$$S(\varphi,\omega) := -\langle \xi_\phi, \log\Delta(\varphi,\omega), \xi_\phi\rangle$$

$$= i \lim_{t\to 0^+} t^{-1}(\varphi([D\omega : D\varphi]_t) - 1)$$

$$S(\rho,\sigma) = \mathrm{Tr}(\rho\log\rho - \rho\log\sigma)$$

Infinite systems

Suppose $\mathfrak{M}$ is an infinite factor, say Type III, and φ a faithful normal state

$$\sup_{(\varphi_i)} \sum p_x S(\varphi_x, \varphi) = \infty$$

where $\varphi = \sum_x p_x \varphi_x$

Better to compare algebras!

Subfactors

A **subfactor** is an inclusion of factors $\mathcal{R} \subset \widehat{\mathcal{R}}$

It is **irreducible** if $\widehat{\mathcal{R}}' \cap \mathcal{R} = \mathbb{C}I$

The **Jones index** $[\widehat{\mathcal{R}} : \mathcal{R}]$ gives the “relative size”

Jones, Invent. Math. **72** (1983)
Kosaki, J. Funct. Anal. **66** (1986)
Longo, Comm. Math. Phys. **126** (1989)

A quantum channel

For finite index inclusion $\mathcal{R} \subset \widehat{\mathcal{R}}$, say *irreducible,*

$$\mathcal{E} : \widehat{\mathcal{R}} \to \mathcal{R}, \qquad \mathcal{E}(X^*X) \geq \frac{1}{[\widehat{\mathcal{R}} : \mathcal{R}]} X^*X$$

quantum channel, describes the **restriction** of operations

Comparing algebras

Want to compare $\widehat{\mathcal{R}}$ and $\mathcal{R}$, with $\mathcal{R} \subset \widehat{\mathcal{R}}$ subfactor

$$
\begin{aligned}
H_\phi(\widehat{\mathcal{R}}|\mathcal{R}) &= \sup_{(\phi_i)} \left(\sum_i [S(p_i\phi_i, \phi) - S(p_i\phi_i \restriction \mathcal{R}, \phi \restriction \mathcal{R})] \right) \\
&= \sup_{(\phi_i)} \underbrace{\left(\chi(\{p_i\}, \{\phi_i\}) - \chi(\{p_i\}, \{\phi_i \restriction \mathcal{R}\})\right)}_{\Delta_\chi}
\end{aligned}
$$

Shirokov & Holevo, arXiv:1608.02203

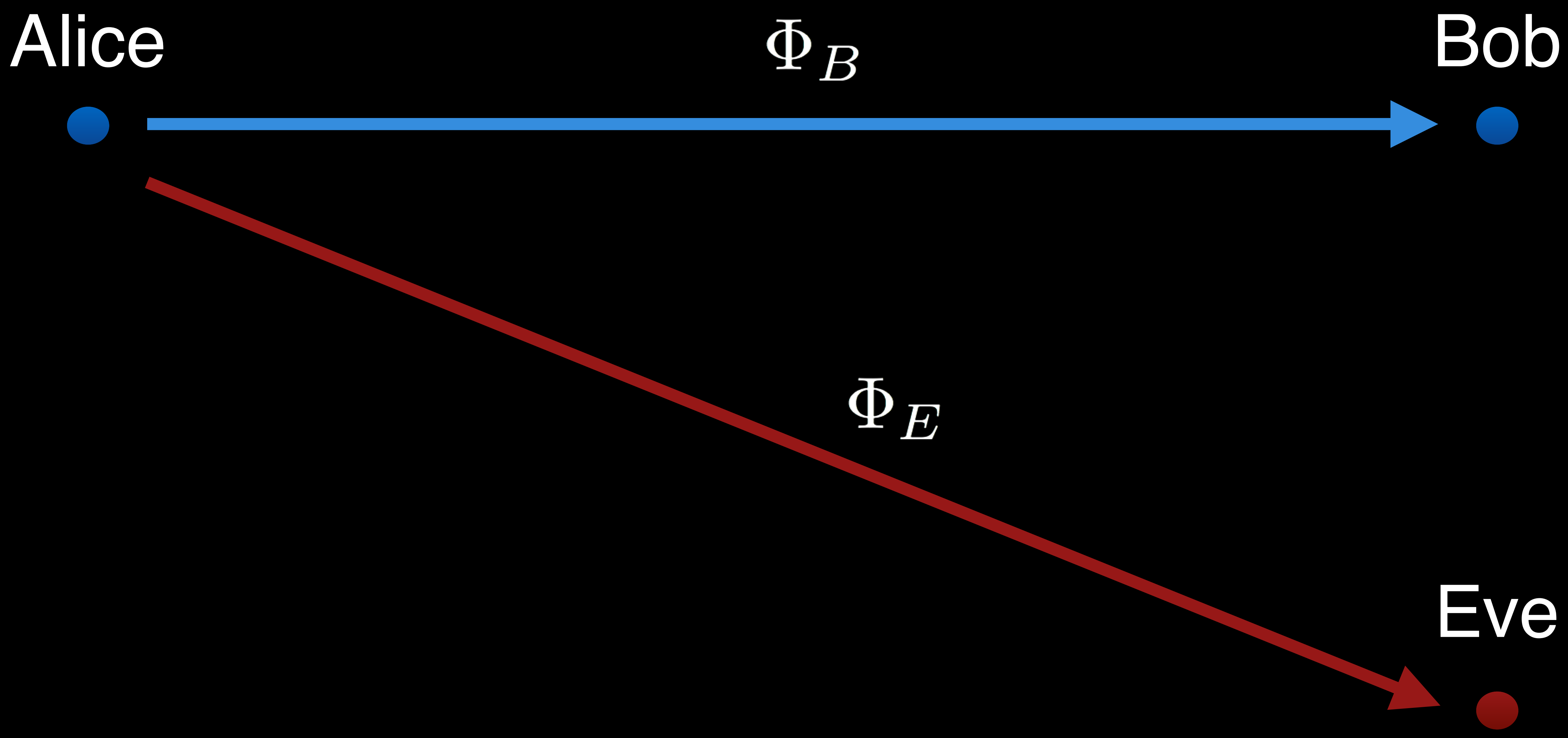
Jones index and entropy

$$\log[\widehat{\mathcal{R}} : \mathcal{R}] = \sup_{\phi:\phi\circ\mathcal{E}=\phi} H_\phi(\widehat{\mathcal{R}}|\mathcal{R})$$

Hiai, J. Operator Theory, ’90; J. Math. Soc. Japan, ‘91

gives an **information-theoretic** interpretation to the Jones index

Quantum wiretapping



Theorem (Devetak, Cai/Winter/Yeung)

The rate of a wiretapping channel is given by

$$\lim_{n\to\infty} \frac{1}{n} \max_{\{p_x,\rho_x\}} \left(\chi(\{p_x\}, \Phi_B^{\otimes n}(\rho_x)\}) - \chi(\{p_x\}, \Phi_E^{\otimes n}(\rho_x)\})\right)$$

A conjecture

The Jones index $[\mathfrak{M} : \mathfrak{N}]$ of a subfactor gives the classical capacity of the wiretapping channel that restricts from $\mathfrak{M}$ to $\mathfrak{N}$.

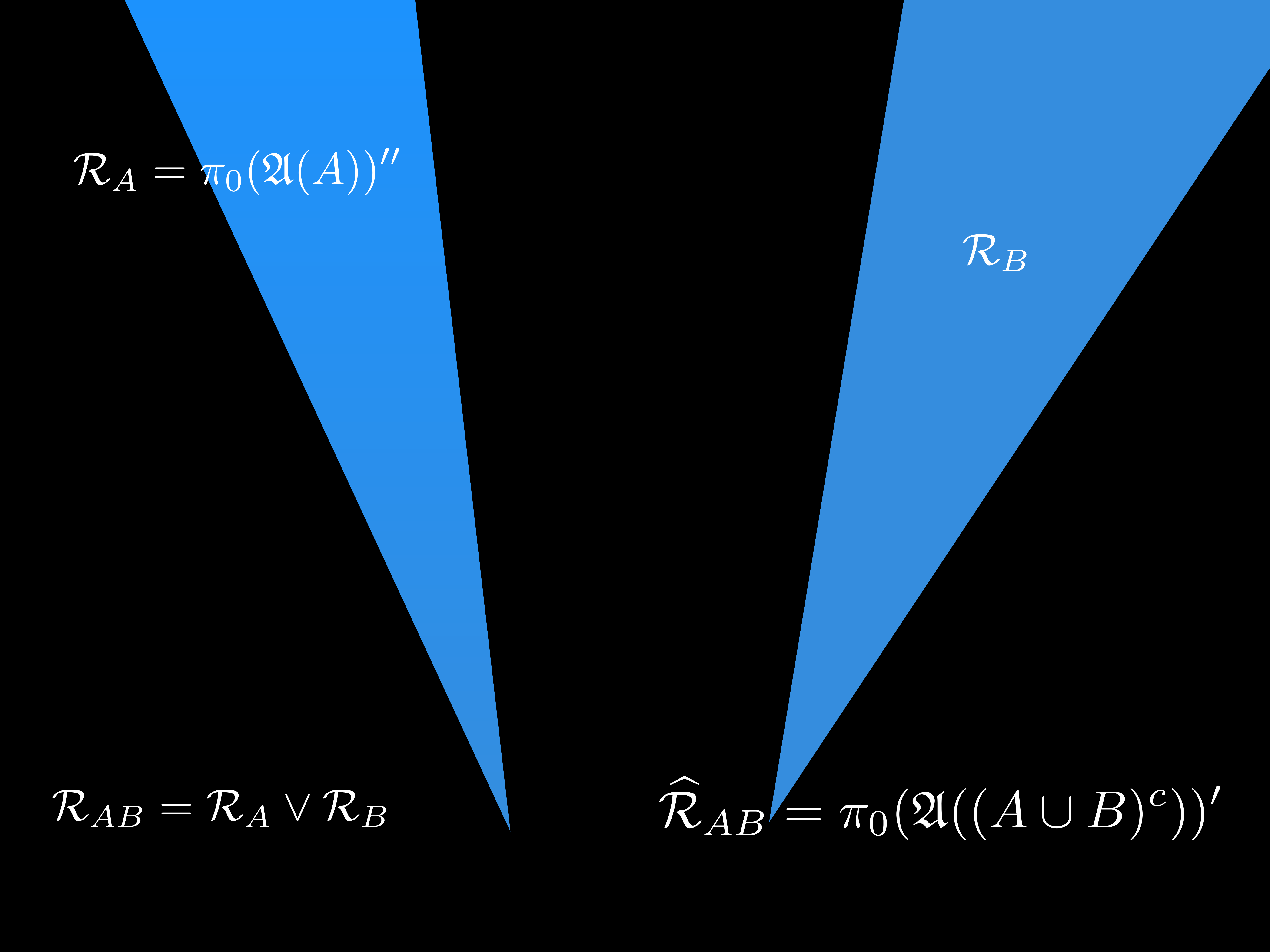
L. Fiedler, PN, T.J. Osborne, New J. Phys **19**:023039 (2017)
PN, Contemp. Math. **717**, pp. 257-279 (2018), arXiv:1704.05562

Some remarks

- use entropy formula by Hiai
- together with properties of the index

$$[\widehat{\mathcal{R}}^{\otimes n} : \mathcal{R}^{\otimes n}] = [\widehat{\mathcal{R}} : \mathcal{R}]^n$$

- averaging drops out: **single letter formula**
- coding theorem is missing


$\mathcal{R}_A = \pi_0(\mathfrak{A}(A))''$
$\mathcal{R}_B$
$\mathcal{R}_{AB} = \mathcal{R}_A \vee \mathcal{R}_B$
$\widehat{\mathcal{R}}_{AB} = \pi_0(\mathfrak{A}((A \cup B)^c))'$

Locality: $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$

but:

$$\mathcal{R}_{AB} \subsetneqq \widehat{\mathcal{R}}_{AB}$$

$$[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}] = \sum_i d_i^2$$