Stability of anyonic superselection sectors

arXiv:1804.03203

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15 August 2019

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This work was funded by the ERC (grant agreement No 648913)

Quantum phases

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution α_t & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.

Quantum phases of ground states

Two ground states ω_0 and ω_1 are said to be *in the same phase* if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that ω_s is a ground state of $H(s)$.

(Chen, Gu, Wen, *Phys. Rev. B* **82**, 2010)

Alternative definition: ω_0 can be transformed into ω_1 with a *finite depth local quantum circuit.*

Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi_\Lambda(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

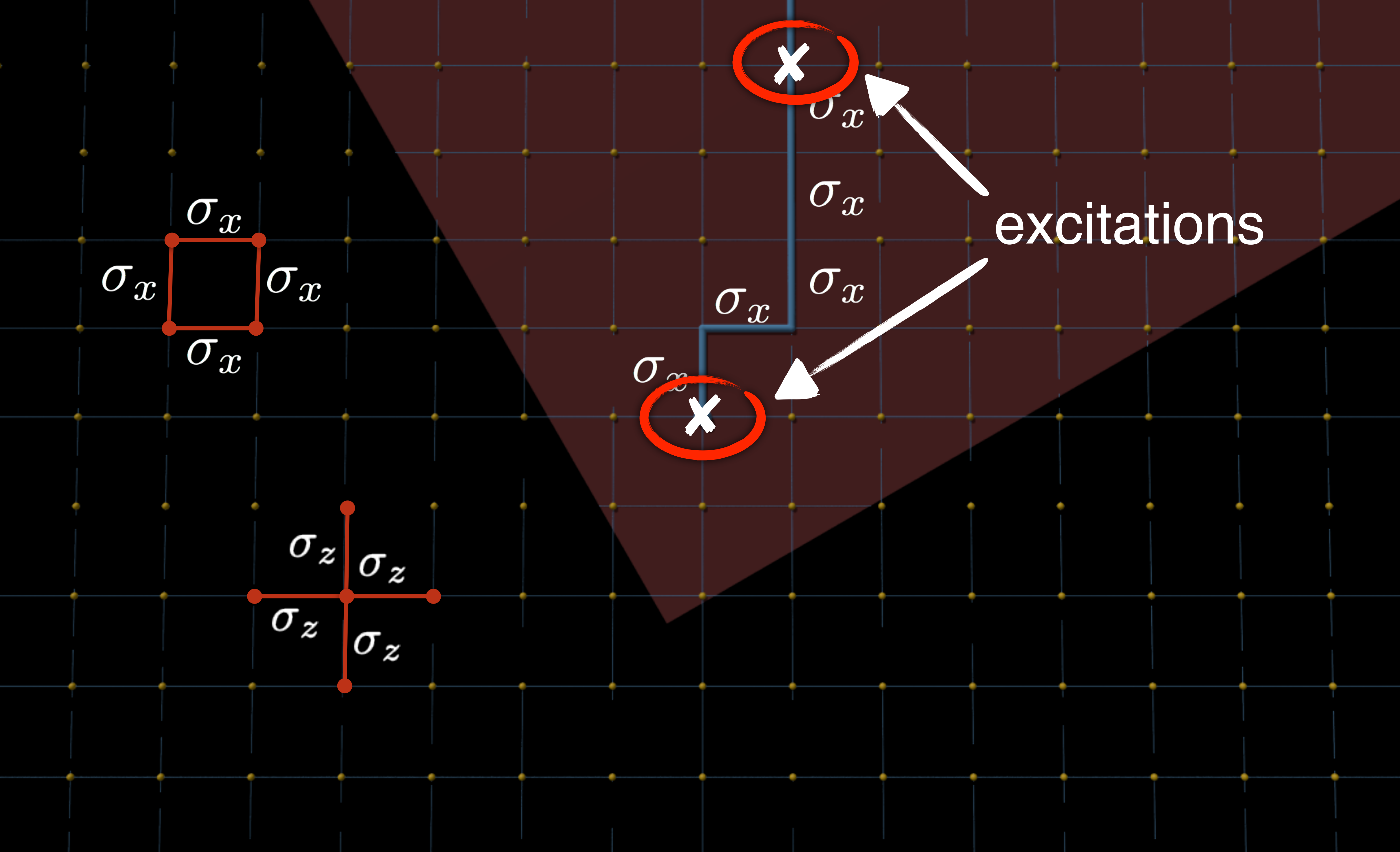
Commun. Math. Phys. **309** (2012)
Moon & Ogata, arXiv:1906:05479 (2019)

Topological phases

Quantum phase outside of Landau theory

- ground space degeneracy
- long range entanglement
- gapped
- anyonic excitations
- modular tensor category / TQFT

Example: toric code



Example: toric code

$\omega_0 \circ \rho$ is a **single excitation** state

$$\rho(A) := \lim_{n \to \infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes observables in presence of **background charge**

Superselection sectors

Localised and transportable morphisms

The endomorphism ρ has the following properties:

- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$
- **transportable:** for Λ' there exists σ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these properties (à la Doplicher-Haag-Roberts)

Doplicher, Haag, Roberts, Fredenhagen, Rehren, Schroer, Fröhlich, Gabbiani, …

Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\mathrm{Rep}\, D(G)$.

Rev. Math. Phys. **23** (2011)
J. Math. Phys. **54** (2013)
Rev. Math. Phys. **27** (2015)

Stability

How much of the structure is invariant?

- Does the gap stay open under small perturbations?
- Is the superselection structure preserved?

Bravyi, Hastings, Michalakis, *J. Math. Phys.* **51** (2010)
Haah, *Commun. Math. Phys.* **342** (2016)

Almost localised endomorphisms

No strict localisation

Technical reason

The superselection criterion is defined on the C*-algebraic level…

… but full analysis requires von Neumann algebras (also, split property, Haag duality for π_0)

For example, **intertwiners** $V \in \pi_0(\mathfrak{A}(\Lambda))''$

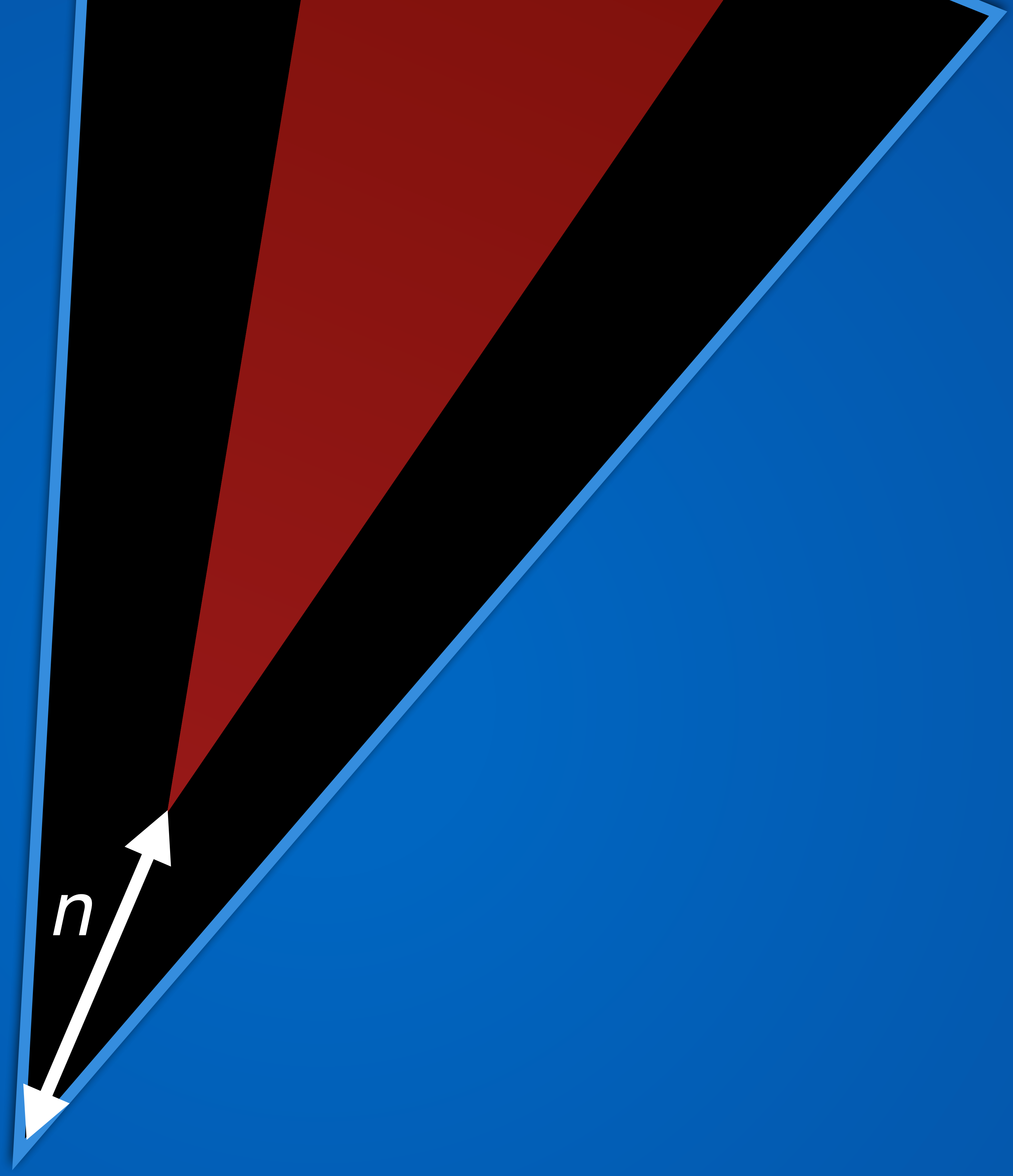
Not clear if/how α_s extends

Almost localised endomorphisms

An endomorphism ρ of $\mathcal{A}$ is called *almost localised* in a cone Λ_α if

$$\sup_{A \in \mathcal{A}(\Lambda^c_{\alpha+\epsilon}+n)} \frac{\|\rho(A) - A\|}{\|A\|} \leq f_\epsilon(n)$$

where $f_\epsilon(n)$ is a non-increasing family of absolutely continuous functions which decay faster than any polynomial in *n*.


n

The semigroup Δ

Define a semigroup Δ of endomorphisms that are

- **almost localised** in cones
- **transportable:** for Λ' there exists σ almost localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$
- intertwiners $(\rho, \sigma)_{\pi_0}$

Can we do sector analysis again?

Stability of Kitaev's quantum double

Asymptotically inner

For general endomorphisms, there are $\{U_n\} \subset \mathfrak{B}(\mathcal{H})$

$$\pi_0(\rho(A)) = \lim_{n\to\infty} U_n \pi_0(A) U_n^*$$

Sequences are not unique, look at such *collections*:

$$\rho(A) = \lim_n U_n A U_n^*,\ \rho'(A) = \lim_n V_n A V_n^*$$

and $R \in (\rho, \rho')_{\pi_0}, \quad R' \in (\sigma, \sigma')_{\pi_0}$ **asymptopia**

$$\lim_{m,n\to\infty} \|[V_n R U_m^*, R']\| = 0$$

Buchholz, Doplicher, Morchio, Roberts & Strocchi. In: *Rigorous quantum field theory* (2007)

Asymptopia

Follow strategy of Buchholz *et al.*: (bi-)**asymptopia**

Using approximate localisation we can get control over the support of $\{U_n\}$

Use this to construct bi-asymptopia and obtain braided tensor category

Buchholz, Doplicher, Morchio, Roberts & Strocchi. In: *Rigorous quantum field theory* (2007)

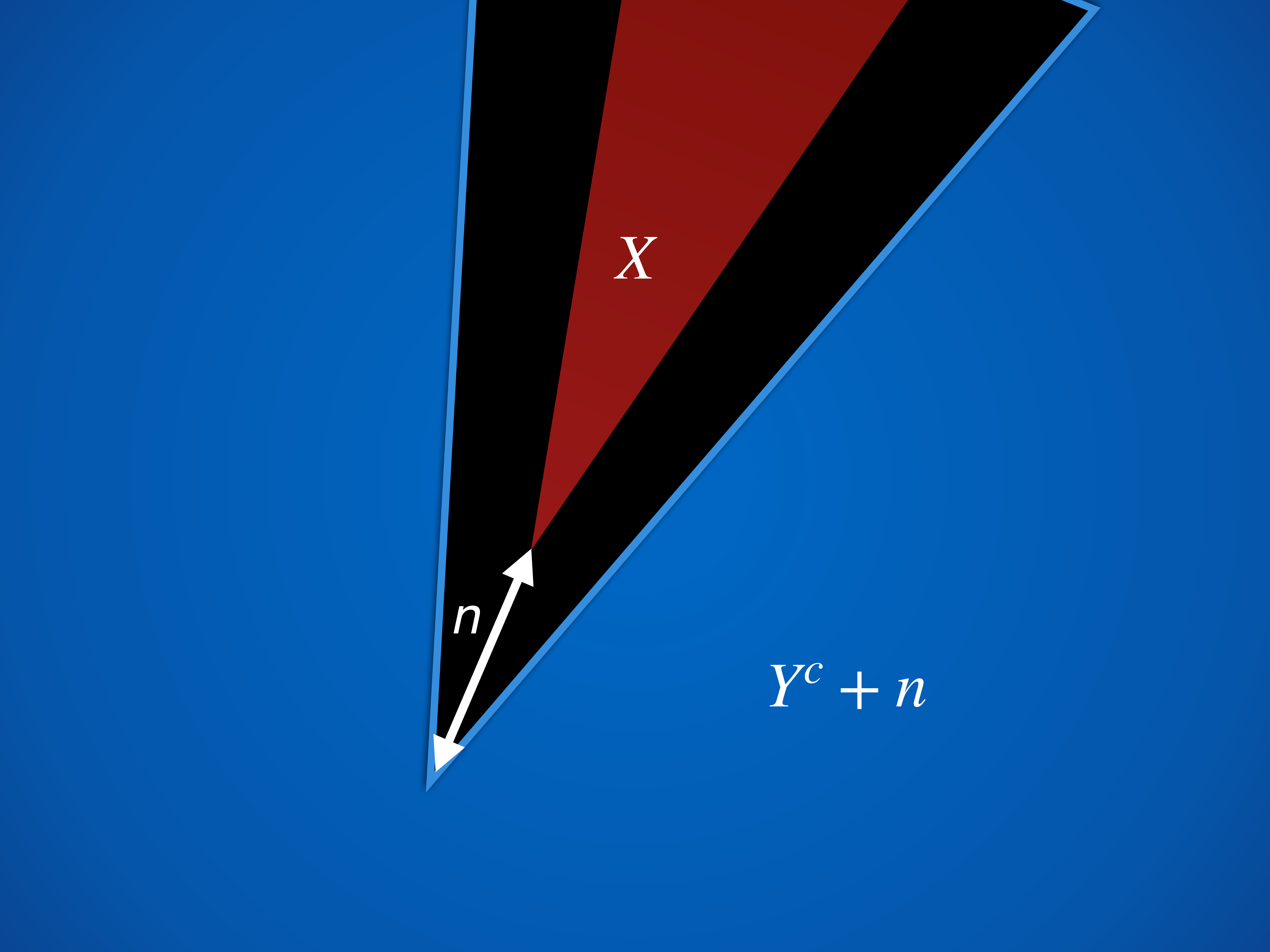
Lieb-Robinson for cones

Quasi-local evolution send observables localised in cones to almost localised observables:

Let X be a cone and Y a cone with a slightly larger opening angle. Then with $A \in \mathfrak{A}(X), B \in \mathfrak{A}(Y^c + n)$

$$\|[\tau_t(A), B]\| \propto \|A\| \|B\| p(d(X, Y + n)) e^{-vt - d(X,Y+n)}$$

Schmitz, Diplomarbeit Albert-Ludwigs-Universität Freiburg (1983)



X
n
$Y^c + n$

Putting it all together

- (bi-)asymptopia give braided tensor category Δ^{qd}
- LR bounds give localisation in cones
- can use this to prove $\Delta^{qd} \cong \alpha_s^{-1} \circ \Delta^{qd} \circ \alpha_s$
- unperturbed model is well understood
- need energy criterion

Theorem

Let G be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each s in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\mathrm{Rep}\, D(G)$.

Cha, PN, Nachtergaele, arXiv:1804.03203